A
MIDDLE SCHOOL
GEOMETRY

PART I

BY
L. G. OWEN, M.A., I.E.S.
AND
K. P. DE, M.A., I.E.S.

407N. Annamalai Kottaiyur

LONGMANS, GREEN AND CO. LTD.

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THIRD EDITION

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PREFACE

The object of this book is to provide a course of Geometry on modern lines suited to the needs of Middle Schools in India and Burma.

In deciding on the sequence of theorems and general method of treatment the authors have, to a large extent, been guided by the recommendations of the Mathematical Association and also by the Report on 'The Teaching of Elementary Geometry' issued by the sub-committee appointed by the Incorporated Association of Assistant Masters in Secondary Schools. Where they venture to differ is in the introduction of Formal Geometry at an earlier stage. The above-mentioned bodies recommend the division of the school course in geometry into three stages. The first stage is mainly experimental. By means of simple practical exercises the pupil becomes acquainted with the common geometrical notions and figures, and discovers certain fundamental facts relating to angles at a point, parallel lines and congruent triangles. No formal written proofs are attempted at this stage. In the second stage the pupil's reasoning powers have become strong enough for the use of the deductive method. Certain fundamental theorems on angles at a point, on parallels and congruence will still be assumed, but other theorems and riders can be derived from these by the deductive process. He learns to prove such theorems and riders and write out proofs. The aim of the third and final stage is to systematize the knowledge of plane geometry already obtained and to arrange the theorems in a logical sequence depending on a comparatively small number of axioms.

The plan adopted in this book is to combine the first and second stages. Wherever possible, a theorem is proved immediately after its discovery by experimental methods. One reason for the adoption of this plan is the fact that in Indian schools pupils begin the study of geometry at a later age than they do in English Schools, an age when their reasoning powers are more fully developed. Another reason is the danger that practical work by itself may develop merely in a series of mechanical exercises in geometrical drawing; 'such work has its usefulness but it is more important that geometry should be utilized, as far as is possible, as a means of training in deductive processes and logical thinking.'[1]

Most Examining Bodies now-a-days do not require for examination purposes formal proofs of some of the earlier theorems; these are marked with an asterisk. To quote from the Report referred to above: 'The truth of these should be established by intuition and experiment, and the teaching of formal geometry should be based on the quasi-axiomatic acceptance of these results. The Committee bases this recommendation upon the fact that the logical basis of the proofs of these theorems, if not insecure, is at any rate very difficult of apprehension by the non-mathematician, while the results appear to nearly all pupils to be obvious.' Nevertheless, the teacher would be well advised not to ignore any theoretical argument which helps to confirm a result obtained by experimental methods. Such arguments may be discussed with the class but need not be written out as formal proofs.

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[1] Report of the Board of Education on 'The Education of the Adolescent.'

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NOTE TO THE TEACHER

The teacher is advised to treat the matter contained in the first chapter in an informal manner as talks with his class. By means of such talks and simple practical exercises the pupil is led to form an idea of the terms used in geometry. He should not, at any rate at this stage, be required to learn by heart the definitions of these terms. The next step consists of exercises with geometrical instruments leading to the discovery of certain fundamental facts which are then enunciated in the form of theorems. For reasons explained in the Preface, formal proofs of these theorems have, wherever possible, been introduced immediately after their discovery by experimental methods. The teacher may, if he chooses, postpone theoretical work till the whole of the practical work has been finished, but *on no account should formal proofs of theorems be taken until the facts stated in these theorems have first been discovered by the pupil himself in his experimental work.*

The pupil should be provided with the following set of geometrical instruments. It would be an advantage if each school had a stock of these which could be lent to the pupil during the geometry lesson.

A Ruler, graduated on one edgein inches and tenths of an inch ; on the other edge in centimetres and millimetres.

A hard pencil.

One or two Set Squares.

A Pair of Compasses.

A Protractor, preferably semi-circular.

CHAPTER I

FUNDAMENTAL IDEAS OF GEOMETRY

SOLID

§ 1. In every-day life when we speak of a solid we mean something distinct from a liquid or a gas. For example, a stone, a block of wood, a lump of ice are all spoken of as solids ; water and oil are liquids ; while the air we breathe is neither a solid nor a liquid, but a gas.

In geometry no such distinction is made. Anything which occupies space, whatever its material may be, is called a solid. We are concerned, not with the composition of the body, but with its shape. Thus a brick, a sheet of paper, water in a jug, the air inside a ball are solids in the geometrical sense.

SURFACE

§ 2. A solid is bounded by a **surface** or surfaces. Surfaces may be either **curved,** such as the surface of an orange, or **flat,** like the surface of a table.

A surface which is flat and not curved is called a **plane.** If **any** two points in a plane be joined by a straight line, the line lies entirely in the surface. This is obviously not true of a curved surface.

Hence to test whether a surface is a plane or not, place the straight edge of a ruler in contact with the surface in different positions. If the whole length of the edge fits exactly on the surface in every position, the surface is a plane.

Some solids, a smooth brick for example, are bounded entirely by plane surfaces. Other solids, such as an egg, a ring, or a bell are bounded by one or more curved surfaces. Others again, such as a rupee or water standing in a cup may be bounded partly by plane and partly by curved surfaces.

Ex. Give other instances of solids bounded (*a*) by plane surfaces, (*b*) by one or more curved surfaces, (*c*) partly by plane and partly by curved surfaces.

It is important to observe that **a surface has no thickness.** A sheet of paper, however thin it may be, is not a surface but a solid, for it occupies space.

LINE

§ 3. Surfaces are, in general, bounded by lines. Lines may be either **straight** or **curved.** Thus the surfaces of a smooth brick are bounded by straight lines, namely, the edges. The flat end of a round pencil is bounded by a curved line.

Ex. Give other instances of surfaces of a solid bounded by (*a*) straight lines, (*b*) curved lines.

Note that **a line has neither breadth nor thickness, only length.** When we draw a line on the blackboard with a piece of chalk, what we see is not a line in the geometrical sense. It is really a solid made up of particles of chalk. It serves, however, to mark the position of the geometrical line.

POINT

§ 4. Lines meet each other in **points.** Thus the edges of a brick are straight lines which meet in points, namely, the corners of the brick. **A point has neither length, breadth nor thickness, that is to say, it has no magnitude or size; all it has is a position or place.** The position of a point is usually denoted by a dot (.) or small cross (+); the latter is the better method. As in the case of a line, what we actually see is not a point in the geometrical sense but a small solid which marks the position of the geometrical point.

DIMENSIONS

§ 5. In order to determine the **volume,** or amount of space, occupied by a brick, three measurements are necessary; we have to measure its length, its breadth and its thickness. It is therefore said to have three **dimensions.**

How many dimensions has a body with a curved surface, an egg for example? We cannot measure its length, breadth and thickness as we can in the case of a brick. The egg, however, occupies space and the amount of space it occupies, that is, its volume, can be **compared** with the volume of the brick. As the brick has three dimensions, so also the egg is said to have three dimensions. Thus **every geometrical solid, whatever its shape may be, is said to be of three dimensions.**

Again, in order to determine the **area,** or amount of surface, of a sheet of note-paper we require two measurements, namely, its length and breadth. Hence, reasoning as before, **every surface, whatever its shape may be, is said to be of two dimensions.**

A line has only one dimension, namely, its length.

A point has no dimensions.

STRAIGHT LINES

§ 6. We have seen that lines may be either straight or curved. In what way do they differ? What properties have straight lines which are not possessed by curved lines? Consider the following questions.

With your ruler draw any two straight lines cutting one another. In how many points do they cut? Can they cut at more than one point? Can you draw two curved lines to cut at more than one point?

Mark any two points on a sheet of paper. With your ruler draw a straight line passing through these two points. Can you draw more than one straight line to pass through these two points? Can you draw more than one curved line to pass through the two points?

Draw two curved lines meeting in two points. They then enclose an area. Can you draw two straight lines to enclose an area?

Suppose there are two roads from a place A to a place B, one being a straight road and the other curved, which road is the shorter of the two?

Consideration of the above questions leads to certain axioms regarding straight lines. An **axiom** is a self-evident truth, that is, a statement which is so obvious that it does not require any proof. We have then the following axioms :—

(1) Two straight lines can cut at one and only one point.

(2) Only one straight line can be drawn to pass through two given points.

(3) Two straight lines cannot enclose an area.

(4) A straight line is the shortest distance between two given points.

Note that the third of these axioms enables you to test whether the edge of your ruler is straight or not. Mark two points A and B on a sheet of paper, the distance between them being slightly less than the length of your ruler. Place the edge of your ruler just below the two points A, B and draw a line through A and B. Now turn your ruler round and place the same edge just **above** the points

A, B and draw another line through A and B. If these two lines do not enclose an area, that is, if they fall on one another, the edge of the ruler is straight.

MEASUREMENT OF STRAIGHT LINES

§ 7. For measuring straight lines it is useful to have a ruler which has one edge divided into inches and tenths of inches and the other edge divided into centimetres and millimetres (tenths of a centimetre). The lengths should be expressed decimally in terms of inches or centimetres Thus write :—

(1) Four and three-tenths inches as 4·3 in. or 4·3″.

(2) Seven-tenths of an inch as 0·7 in. or 0·7″.

(3) Six centimetres nine millimetres as 6·9 cm.

(4) Eight millimetres as 0·8 cm. (or 8 mm).

Consider the following figure which represents a line AB measured by a ruler which is divided into inches and tenths of inches.

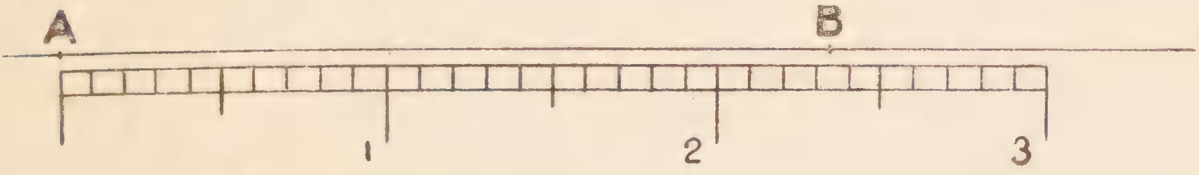


FIG. 1.

The point A is opposite the zero division of the ruler. The point B lies between the divisions 2·3 and 2·4 but is nearer the former. Hence, to the nearest tenth of an inch, we should say that the length of AB is 2·3 inches.

If we imagine the distance between the divisions 2·3 and 2·4 divided into 10 equal parts, that is, into hundredths of an inch, we see that the point B is, as near as we can judge, 4 hundredths of an inch beyond the division 2·3. Hence, to the nearest hundredth of an inch, we should say that the length of AB is 2·34 inches.

EXERCISE 1

1. Measure the lengths of AB and CD (i) in inches and tenths of an inch, (ii) in centimetres and millimetres.

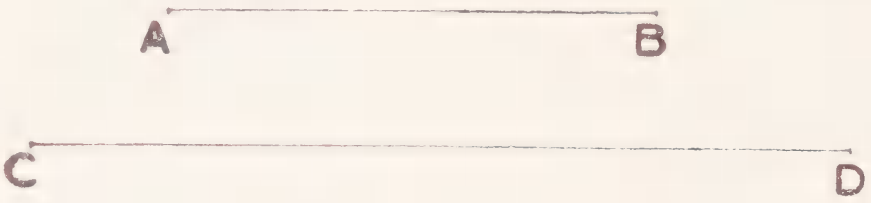


FIG. 2.

2. Measure AB and BC in inches and tenths of an inch and test your result by measuring AC.

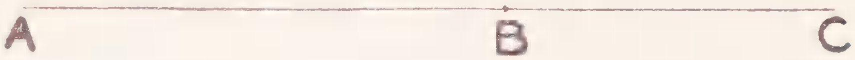


FIG. 3.

3. Repeat Ex. 2 using centimetres instead of inches.

4. Measure AB and AC in inches and tenths of an inch. Test your result by measuring BC.

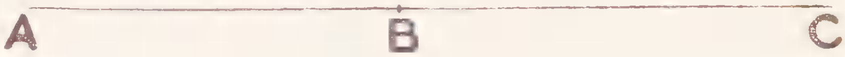


FIG. 4.

5. Repeat Ex. 4, using centimetres instead of inches.

6. Measure AB, BC, CD, DE in inches, tenths of an inch, and, as nearly as you can, in hundredths of an inch. Add the measurements and test your result by measuring AE.

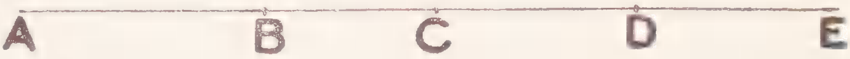


FIG. 5.

7. Repeat Ex. 6, using centimetres and millimetres.

8. Draw straight lines of the following lengths: 3·4 in., 6·8 cm., 0·9 in., 4·5 cm., 0·6 cm., 2·1 in., 7 mm., 46 mm.

9. Draw a straight line 6 in. long. Measure it in centimetres, giving the result to the nearest millimetre. Divide the result by 6 and thus find how many centimetres there are in an inch.

10. Repeat Ex. 9 by drawing straight lines of length 4 in. and 3 in. Compare your results. Which should give the more accurate result?

11. A centimetre is one hundredth part of a metre. Draw a straight line 10 cm. long and measure it in inches. Hence find how many inches there are in a metre.

12. Mark on your paper any two points A, B. Guess the length of the straight line joining A and B. Test your guess by measurement with a ruler. Repeat this exercise several times measuring both in inches and centimetres.

13. Draw any straight line AB. Guess the position of the middle point of the line, marking it by a short cross-line. Test your guess by measuring the two parts of the line.

When a point divides a straight line into two equal parts, the straight line is said to be *bisected* at that point.

THE CIRCLE

§ 8. Place the steel point of your compasses at some point O on a flat sheet of paper and, keeping the span of the compasses unaltered, let the pencil point trace out a curved line. The figure enclosed by the curved line is called a **circle.** The curved line itself is called the **circumference** of the circle. The point O is called the **centre** of the circle.

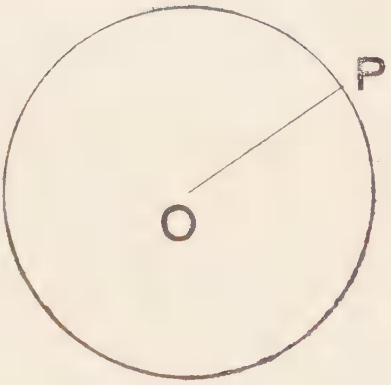


FIG. 6.

The term circle is used in two different senses. It really means the figure enclosed by the circumference, but it is often taken to mean the circumference itself. Thus when we speak of points on a circle we mean points on the circumference of the circle.

It is clear that all points on the circumference, such as P, are at the same distance from the centre. Any line such as OP drawn from the centre to a point on the circumference is called a **radius.** Hence **the radii of a circle are all equal.**

It is important to note that if the flat surface on which the circle is drawn is bent or folded the figure ceases to be a circle.

For in that case all the points on the curved line may not be at equal distances from the point O. For instance, the sheet of paper can be folded in such a way that the point P lies on the point O. Hence a circle is a plane figure, that is, all points on it must lie on a plane or flat surface.

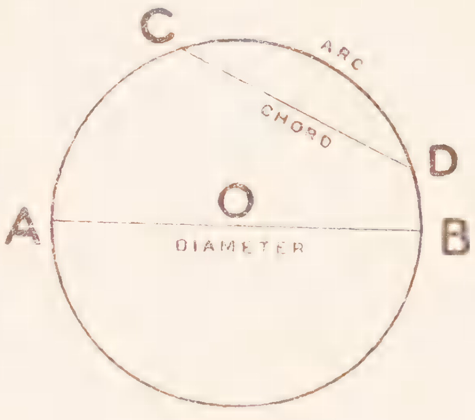


FIG. 6*a*.

A straight line joining any two points on the circumference, such as CD, is called a **chord.**

When the chord passes through the centre, it is called a **diameter.** A diameter is therefore equal to twice the radius.

Any portion of the circumference is called an **arc.**

It is clear that a diameter divides the circle into two equal parts. Each part is called a **semi-circle.**

EXERCISE 2

1. What is the greatest number of points at which a straight line can cut a circle ?

2 What is the greatest number of points at which two circles can cut one another ?

3. Will two circles which have the same centre but different radii cut one another ?

4. Draw a circle of radius 1·8 in. and place in it a chord of length 2·1 in.

What is the length of the longest chord which can be placed in the circle ?

5. Draw a straight line AB of length 3·2 in. With A as centre draw a circle of radius 1·6 in., and with B as centre draw a circle of radius 1·3 in.

Why do the two circles not cut one another ?

6. Draw a straight line AB of length 4·8 cm. With centre A and radius 3·0 cm. draw a circle, and with centre B and radius 2·5 cm. draw another circle.

If C and D are the two points at which those circles cut, measure the distance of each of the points C and D from A and from B.

7. Draw a straight line AB 6·4 cm. long. Find, with your compasses, a point which is 4·6 cm. from A and 3·9 cm. from B.

How many such points are there?

8. Draw a straight line 2·4 in. long and find a point which is 1·8 in. from each end.

How many such points are there?

9. Draw a straight line AB 4 in. long. Can you find a point which is 1·9 in. from A and 1·6 in. from B? Why not?

ANGLES

§ 9. Suppose a straight line OA is turned about a point O from the position OA to the position OB. The straight line is now pointing in a different **direction** and is said to have turned through an **angle.** Hence the angle between the straight lines OA and OB may be considered as measuring the **amount of turning** which is required to bring a straight line from the position OA to the position OB.

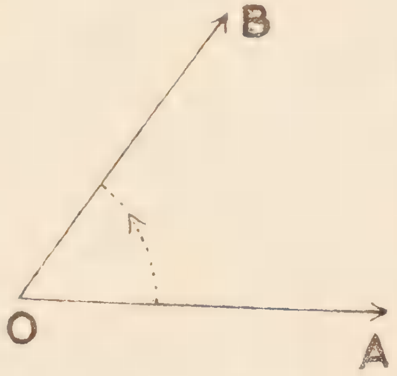


FIG. 7.

It may be regarded also as measuring the **change of direction** of the straight lines OA and OB.

A straight line which turns in the same direction as that in which the hands of a clock turn is said to be turning **clockwise.** When it turns in the opposite direction, it is said to be turning **counter-clockwise.**

OA and OB are called the **arms** of the angle, and the point O where they meet is called the **vertex** of the angle. Note that the size of an angle has nothing to do with the length of its arms.

The angle between the straight lines OA and OB may be named in various ways. It may be called either the angle AOB or the angle BOA, where in each case the

ROJA MUTHIA

letter for the vertex is placed in the middle. When no other angle has its vertex at O the angle may be called simply the angle O. It is often convenient to name an angle by means of a letter placed within the arms near the vertex, such as the angle **a** and the angle **b** in Fig. 8.

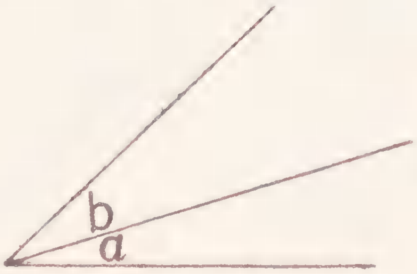


FIG. 8.

The abbreviation for 'the angle AOB' is $\angle$ AOB or $\widehat{\text{AOB}}$.

When two angles lie on different sides of a common arm, they are said to be **adjacent** angles. Thus in Fig. 9, $\angle$ AOB and $\angle$ BOC are adjacent angles, the common arm being OB.

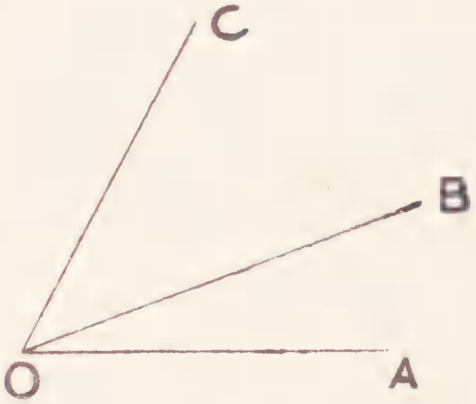


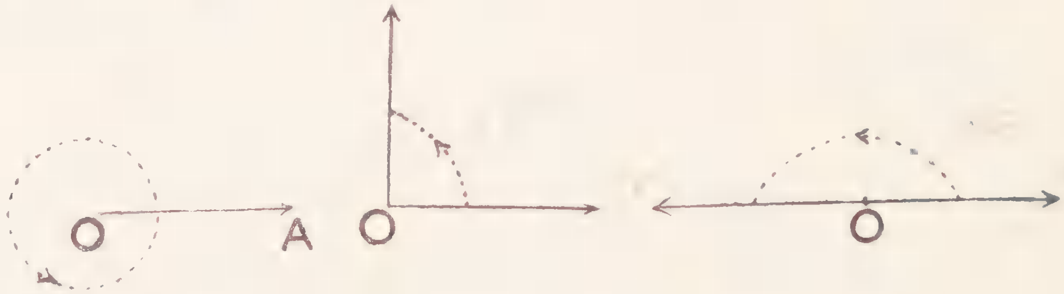
FIG. 9.

It is clear that $\angle \text{AOC} = \angle \text{AOB} + \angle \text{BOC}$.

When a straight line OA makes a complete turn about the point O, it returns to its original position. One quarter of a complete turn is called a **right angle.**

A complete turn is therefore 4 right angles.

Half a complete turn is 2 right angles and is called a **straight angle**.



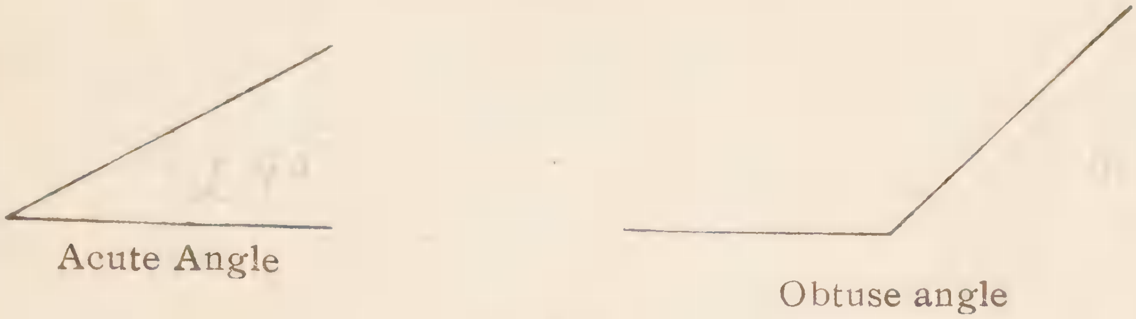
Complete turn Right angle Straight angle

FIG. 10.

An angle which is smaller than a right angle is called an **acute angle.**

Refl

An angle which is greater than a right angle and smaller than a straight angle is called an **obtuse angle.**



Acute Angle

Obtuse angle

FIG. 11.

When one straight line makes a right angle with another straight line the two lines are said to be at right angles or **perpendicular** to each other.

When a straight line divides an angle into two equal parts it is said to **bisect** the angle, and the straight line is called the bisector of the angle.

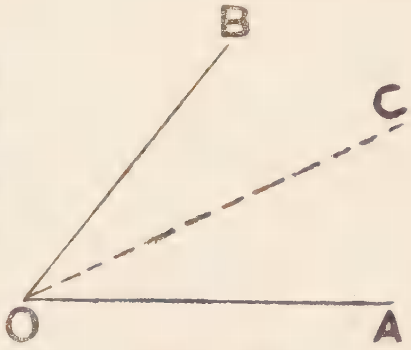


FIG. 12.

Thus, in Fig. 12, if ∠ AOC = ∠ BOC, then ∠ AOB is bisected by the straight line OC, and OC is the bisector.

EXERCISE 3

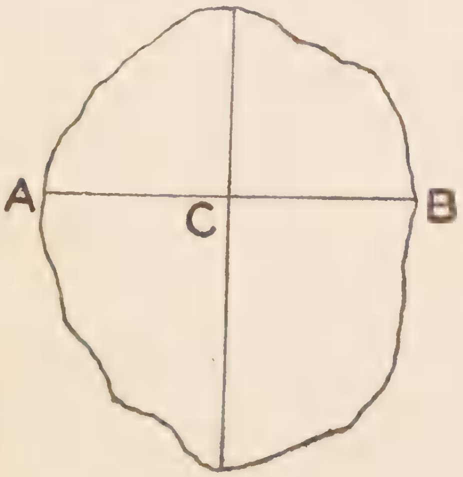


FIG 13.

✓1. Take any piece of paper and fold it, as in Fig. 13, about a straight line, AB say. Fold it again in such a way that a part of AB, AC say, falls on the remaining part CB. Now unfold the paper. What are the angles formed by the creases at C ? ✓

Polygon

2. Cut out a piece of paper having an angle **AOB** as in Fig. 14. Fold the paper so that the edge **OA** falls on the edge **OB**. Now unfold the paper. It is clear that the straight line **OC** formed by the crease **bisects** the angle **AOB**.

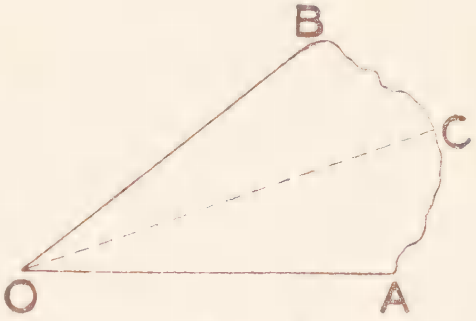


FIG. 14.

3. How long does it take (*a*) the minute-hand, (*b*) the hour-hand of a clock to turn through one right angle?

4. Through how many right angles does the minute-hand of a clock turn in 1 hour, $\frac{1}{4}$ hour, 7 hours, 1 day?

5. Through what fraction of a right angle does the hour-hand of a clock turn in 1 hour, 15 minutes, 10 minutes?

6. What fractions of a right angle do the hands of a clock make with each other at the following times: 9, 2, 4 and 11 o'clock?

State in each case whether the angle is right, acute or obtuse.

7. A wheel makes 30 complete turns in a minute; through how many right angles does it turn in a second?

8. The Earth makes a complete turn in 24 hours. In what time does it turn through (*a*) a right angle, (*b*) a straight angle?

MEASUREMENT OF ANGLES

§ 10. We have already defined a right angle as a quarter of the complete turn made by a straight line when it is turned about a point fixed in the line. It is clear that **all right angles are equal.** We can therefore take a right angle as the standard angle with which to compare other angles.

For purposes of measurement a right angle is divided into 90 equal parts called **degrees.** Thus one right angle is equal to 90 degrees, which is written 90°. Two right angles = 180°; 4 right angles = 360°; half a right angle = 45°; and so on.

For measuring angles we have an instrument known as the **protractor.**

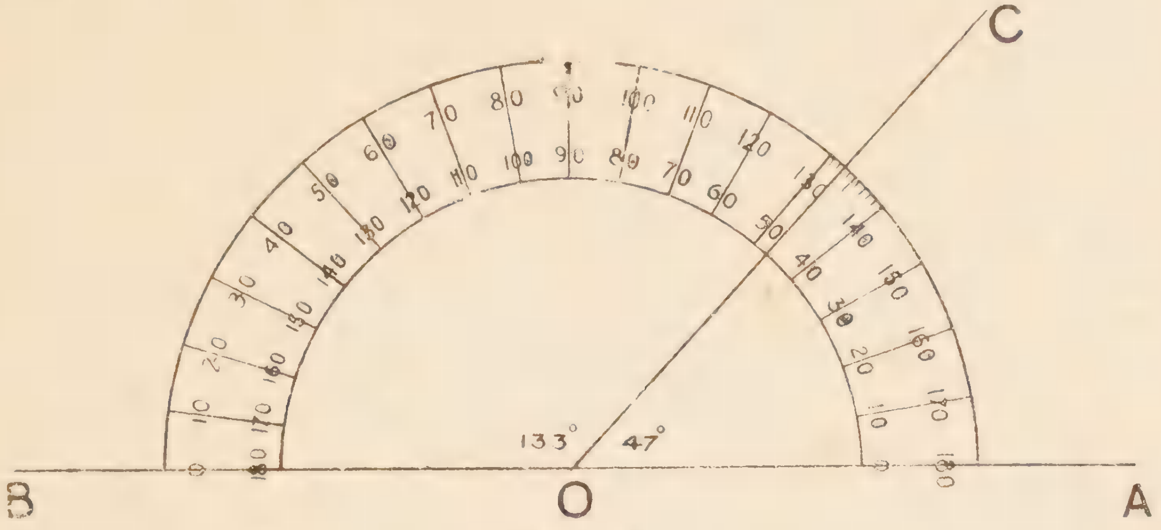


FIG. 15.

To measure ∠ AOC (Fig. 15), place the centre of the protractor at O and the base line (that is, the edge which passes through the centre and a zero mark) along OA. Which of the two sets of figures should be used? AOC is obviously an acute angle, that is, less than 90°; the inner set of figures must therefore be used. The angle is 3° away from 50° but is less than 50°. Therefore the correct measurement of ∠ AOC is 47°.

Similarly with ∠ BOC. This angle is obviously obtuse, that is, greater than 90°. Hence the outer set of figures must be used. The angle is 3° away from 130° but is greater than 130°. Therefore the correct measurement of ∠ BOC is 133°.

When the angle has been measured, write the number of degrees within the angle as shown in Fig. 15.

In order to measure an angle it is sometimes necessary to **produce** the arms, that is, make them longer.

Mistakes in reading the protractor are common with beginners. The following idea may be found useful :—Imagine that the angle is formed by turning a straight line about the point O. In the case of ∠ AOC the angle is formed by turning the straight line from the position OA to the position OC. It must start at 0° and increase in value; we therefore use the inner set of figures. In the case of ∠BOC the angle is formed by turning a straight line from OB to OC. Again it must start 0° and increase in value; we therefore use the outer set of figures.

To construct with the protractor an angle of given measure. Suppose at a point A in a straight line AB it is required to draw a straight line making with AB a given angle, say 64°. Place the centre of the protractor at A and the base line along AB. Then opposite the mark which denotes 64° make a small dot with your pencil and join this dot to the point A. Write 64° within the angle.

For drawing right angles set squares are more convenient than the protractor.

To draw through a given point P a straight line perpendicular to a given straight line AB.

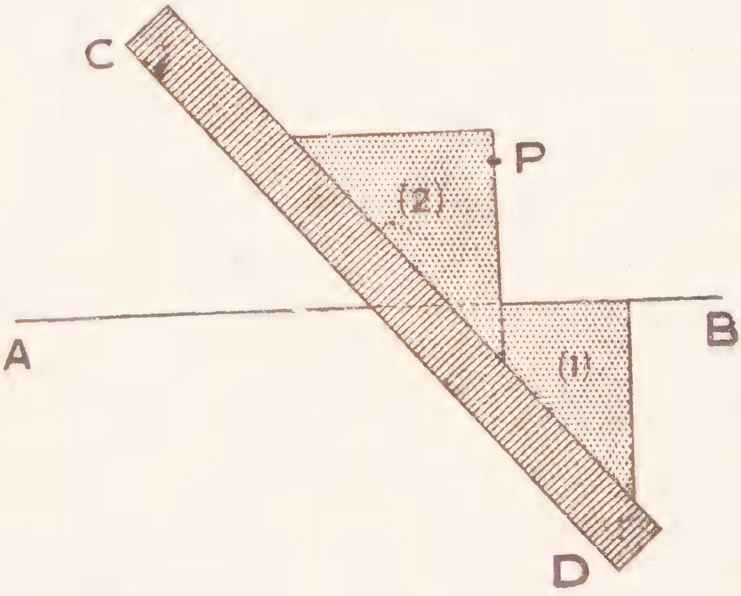


FIG. 16.

Place the set square in the position (1) with one of the edges which contains the right angle along the line AB, as shown in Fig. 16. Then place a ruler CD with one of its edges along the edge of the set square which is opposite the right angle. Holding the ruler firmly, slide the set square along it from the position (1) to the position (2), and use the edge to draw through P a perpendicular to AB.

The length of the perpendicular drawn from a point P to a straight line AB is called the **perpendicular distance,** or simply the **distance,** of P from AB.

EXERCISE 4

1. How many degrees are there in $1\frac{1}{2}$ right angle, $\frac{2}{3}$ right angle, $\frac{4}{5}$ right angle?

2. What fractions of a right angle are 30°, 54°, 75°, 80°?

3. Through how many degrees does (*a*) the minute-hand, (*b*) the hour-hand of a clock turn in one hour?

4 Name, by means of three letters, each of the angles marked 1 to 13 in Fig. 17

5. How many pairs of adjacent angles are there in Fig. 17? Name them.

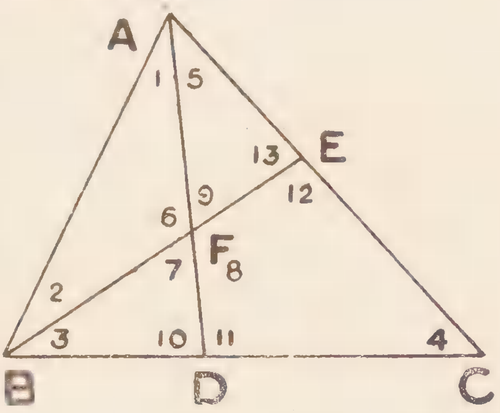


FIG. 17.

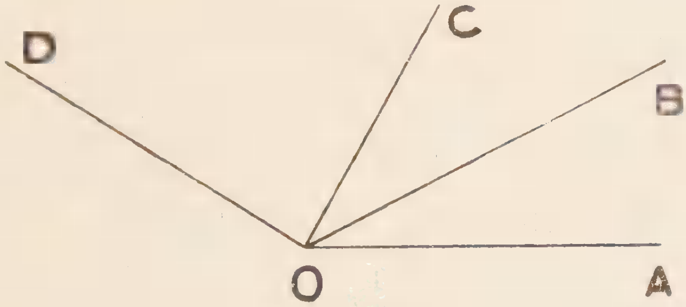


FIG. 18

6. Draw four straight lines meeting in a point as in Fig. 18. Measure with the protractor the angles **AOB**, **BOC**, **COD**. Find their sum and check the result by measuring ∠ **AOD**.

7. Draw any other four straight lines like the above. Measure the angles **AOD**, **COD**, **BOC** and state whether they are acute or obtuse. Check your results by measuring ∠ **AOB**.

8. Draw a figure like Fig. 19 but of a larger size. Measure the angles and find the sum of the angles

(*a*) **AEB** and **BEC**.
(*b*) **BEC**, **ECB** and **CBE**.
(*c*) **AEB**, **EBA** and **BAE**.
(*d*) **ADC**, **DCA** and **CAD**.
(*e*) **ADC**, **DCB**, **CBA** and **BAD**.

9. Draw angles of 40°, 168°, 71°, 13°, 92°, 88°, 135°. State whether they are acute or obtuse.

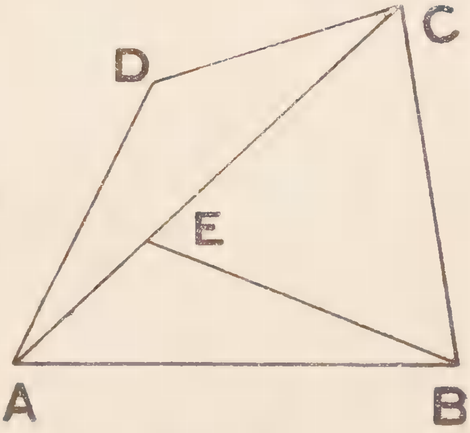


FIG. 19.

10. Draw freehand, as near as you can guess, angles of 90°, 45°, 30°, 60°, 135°, 120°. Check your guess by measurement with the protractor.

11. Draw an angle BAC = 75°, making AB = 6·8 cm. and AC = 8·1 cm. Measure the side BC and the angles B and C.

12. Draw a straight line AB = 3·7 in. At A draw a straight line making with AB an angle of 32°, and at B a straight line making with BA an angle of 57°. Let these lines meet at C. Measure the angle C and the sides AC, BC.

13. Draw a circle of any radius and take any diameter AB of the circle. Join A, B to any point P on the circumference. Measure ∠ APB.

Repeat this exercise, taking different positions of P. What fact do you infer?

14. Draw a straight line AB = 7·2 cm. Mark a point C in it such that AC = 3·3 cm. With your set squares draw a straight line CD perpendicular to AB making CD = 5·5 cm. Join AD, BD and measure the angles ABD, BDA and DAB.

15. Draw a straight line AB = 2·4 in. Draw angles BAC = 28° and ABC = 137°. Using your set squares draw a straight line from C perpendicular to AB (produced). Measure the length of this perpendicular

16. Draw any angle and guess the position of the bisector of the angle, making a freehand drawing of it. Test your guess by measuring with the protractor the two parts into which the angle has been divided.

Repeat this exercise with the angles of different sizes, acute and obtuse.

CHAPTER II

ANGLES AT A POINT

§ 1. Through any point **O** draw three straight lines **OA, OB, OC** as in Fig. 20. Measure with your protractor the angle between each pair of consecutive lines, that is, between each line and the next line. Find the sum of these angles.

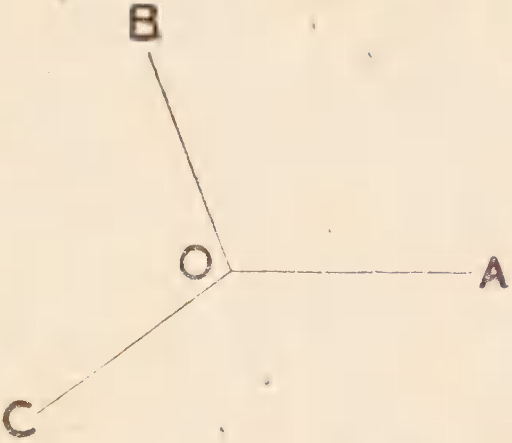


FIG. 20.

Repeat the exercise with four or more straight lines drawn from a point in any directions. From your results you find that the sum of the angles appears to be about 360°.

It can be shown, however, without making any measurements at all, that the sum of these angles must be exactly 360° or four right angles.

Consider the angles formed by four straight lines drawn from a point **O**, as in Fig. 21. ∠ **AOB** is formed by turning a straight line about **O** in a counter-clockwise direction from **OA** to **OB**, ∠ **BOC** by turning the line from **OB** to **OC**, ∠ **COD** from **OC** to **OD**, and ∠ **DOA** from **OD** to **OA**. The line has now returned to its first position and has therefore made a complete turn. Hence the sum of the angles **AOB, BOC, COD, DOA** is 4 right angles.

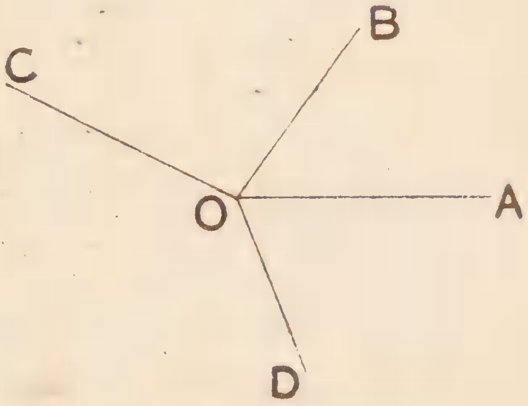


FIG. 21.

The proof is clearly the same for any number of lines drawn from the point O.

A geometrical fact proved in this manner by a process of reasoning is called a **theorem.**

The above fact can be stated in a general form as follows :—

* THEOREM 1.—**If any number of straight lines meet in a point, the sum of the angles between consecutive lines is four right angles.**

§ 2. Draw any straight line CD standing on another straight line AB as in Fig. 22. Measure the adjacent angles ACD BCD, (this can be done without moving the protractor) and find their sum. It is obvious that in all such cases the sum of the two angles is 180° or two right angles.

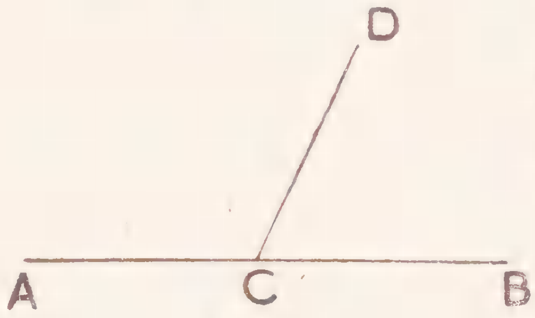


FIG. 22.

Here again, we could have arrived at this fact without making any measurements. For, ∠ BCD is formed by turning a straight line about C in a counter-clockwise direction from CB to CD, and ∠ DCA is formed by turning the line from CD to CA. The line has now made half a complete turn. Hence the sum of the angles is two right angles.

This geometrical fact can be stated in a general form as follows :—

* THEOREM 2.—**When one straight line stands on another straight line the sum of the adjacent angles is two right angles.**

When the sum of two angles is two right angles, they are said to be **supplementary angles,** and each angle is said to be the **supplement** of the other. Thus 140° and

40° are supplementary angles ; 68° is the supplement of 112°. The supplement of a right angle is a right angle.

When the sum of two angles is a right angle they are said to be **complementary angles,** and each angle is said to be the **complement** of the other. Thus 30° and 60° are complementary angles ; 54° is the complement of 36°.

EXERCISE 5

1. OA, OB, OC, OD, OE are five straight lines meeting at a point O.

(i) If ∠ AOB = 59°, ∠ BOC = 78°, ∠ COD = 134°, ∠ DOE = 15° ; what is the ∠ EOA ?

(ii) If ∠ AOB = 100°, ∠ BOC = 113°, ∠ DOE = 45°, ∠ EOA = 88°, what is ∠ COD ?

In each case test your result by constructing the angles.

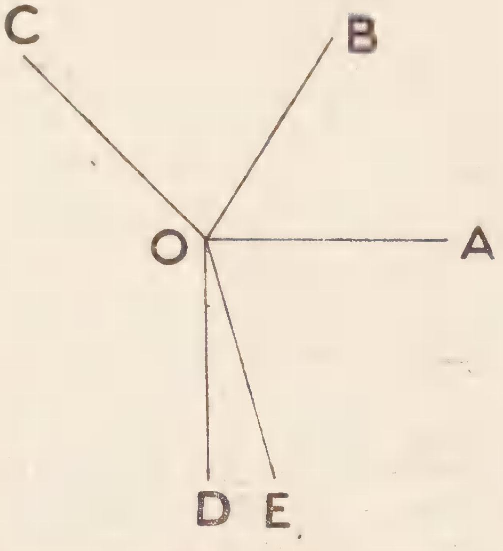


FIG. 23.

2. Write down the supplements of the following angles : 30°, 45°, 120°, 66°, 108°, 117°.

3. Write down the complements of the following angles : 45°, 60°, 80°, 15°, 39°, 72°, 90°.

4. Draw any straight line AB and take any point O in it ; draw from O any straight lines OC, OD on the same side of AB. Measure the angles AOC, COD, DOB and find their sum.

By which theorem can you check your result ?

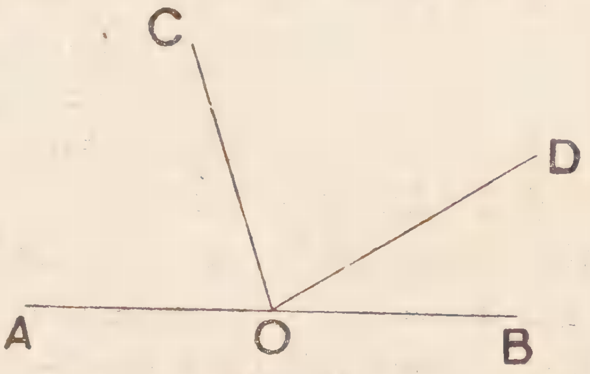


FIG 24.

5. In Fig. 24 :

(i) if ∠ AOC = 37°, ∠ COD = 78°, calculate ∠ DOB ;

(ii) if ∠ AOC = 95°, ∠ DOB = 32°, calculate ∠ COD.

In each case verify your result by measurement.

6. Draw a straight line AB and take any point O in it. Draw from O any number of straight lines on the same side of AB. What is the sum of all the angles formed by consecutive lines ?

ROJA MUTHI

7. In Fig. 25, four straight lines are drawn from O so that $\angle AOB = 42^\circ$, $\angle BOC = 85^\circ$, $\angle AOD = 145^\circ$. If AO is produced to A′, calculate the value of $\angle COA'$ and $\angle DOA'$.

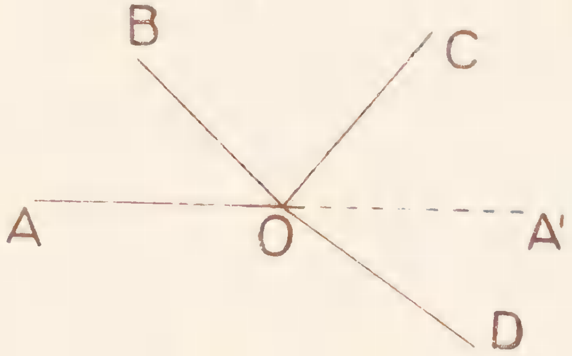


FIG. 25.

8. In Fig. 26, CD is any straight line standing on another straight line AB. CE is the bisector of $\angle ACD$ and CF is the bisector of $\angle BCD$.

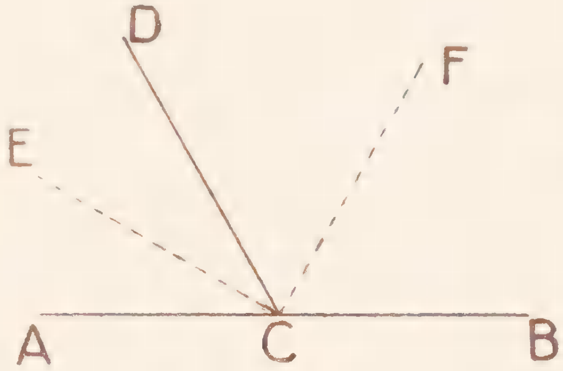


FIG. 26.

Prove, by means of Theorem 2 that CE and CF are at right angles.

[A geometrical exercise of this kind which is to be proved theoretically is called a **rider.** As an example, we shall show how the proof should be written out.

Given a straight line CD standing on a straight line AB and the angles ACD, BCD bisected by the lines CE, CF respectively.

To **prove** that $\angle ECF$ is a right angle.

Proof. $\angle ECD = \frac{1}{2} \angle ACD$.

$\angle FCD = \frac{1}{2} \angle BCD$.

$\therefore \angle ECD + \angle FCD = \frac{1}{2}(\angle ACD + \angle BCD)$.

$= \frac{1}{2}$ of 2 right angles.

$=$ 1 right angle.

$\therefore \angle ECF$ is a right angle.]

§ 3. Draw an $\angle AOB = 50^\circ$. Then draw at O on the other side of OB an $\angle BOC = 130^\circ$. These two angles are adjacent angles. OB is their common arm and OA, OC are their exterior or outside arms.

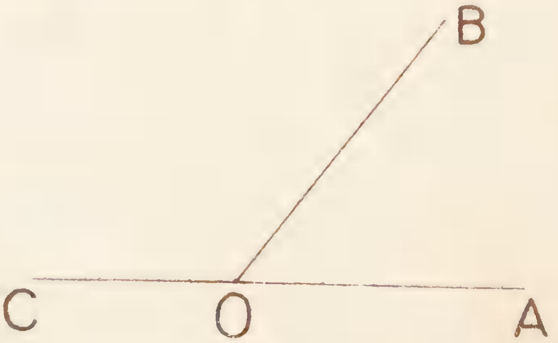


FIG. 27.

What is the sum of these two angles? Are the exterior arms OA, OC in the same straight line?

Repeat the exercise with the following pairs of adjacent angles, giving in each case the sum of the angles and stating whether the exterior arms are in the same straight line or not.

(i) $\angle AOB = 116°$, $\angle BOC = 64°$.
(ii) $\angle AOB = 82°$, $\angle BOC = 94°$.
(iii) $\angle AOB = 47°$, $\angle BOC = 138°$.
(iv) $\angle AOB = 24°$, $\angle BOC = 156°$.
(v) $\angle AOB = 98°$, $\angle BOC = 82°$.

From these exercises we find that when the sum of the two angles is 180° the exterior arms are in the same straight line. This is, of course, only what we should expect to find. For, suppose a straight line is turned about the point O from the position OA to the position OB, and then from the position OB to the position OC. Since the sum of the angles AOB, BOC is two right angles, the line has made half a complete turn. Therefore OC is in the direction of AO produced.

This fact can be stated in a general form as follows:—

***THEOREM 3.—If the sum of two adjacent angles is two right angles the two exterior arms are in the same straight line.**

EXERCISE 6

1. $\angle AOB = 45°$, $\angle BOC = 72°$, $\angle COD = 63°$. Are OA, OD in the same straight line?

2. Five straight lines OA, OB, OC, OD, OE, meet in a point O, such that $\angle AOB + \angle BOC + \angle COD + \angle DOE = 2$ right angles; prove that OA and OE are in the same straight line.

3. From a point O in a straight line AB a straight line OC is drawn perpendicular to AB on one side of it, and another straight

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line OD is drawn perpendicular to AB on the other side. Prove that OC, OD are in the same straight line.

4. From a point O in a straight line AB draw a straight line OC making ∠BOC = 42°. On the other side of AB draw OD making ∠AOD = 42°.

How do you know, without actually measuring the angle, the number of degrees in ∠AOC?

What then is the sum of the angles BOC, BOD?

Are OC, OD in the same straight line? Why?

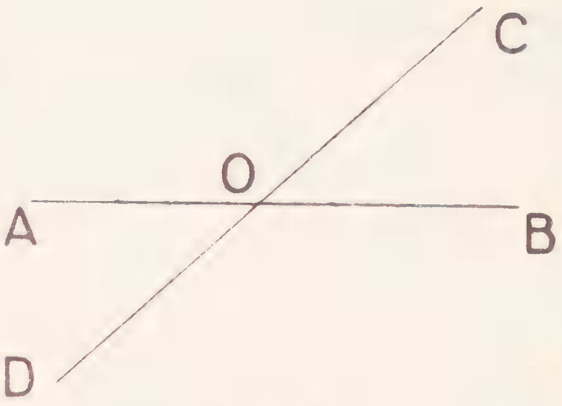

FIG. 28.

5. O is any point in a straight line PQ. Equal angles QOR, POS are drawn on opposite sides of PQ. Prove, by means of Theorems 2 and 3, that OR and OS are in the same straight line.

§ 4. Draw any two straight lines intersecting, that is, cutting each other, at a point O. The angles AOC, BOD are said to be a pair of **vertically opposite angles.** Also the angles BOC, AOD are another pair of vertically opposite angles.

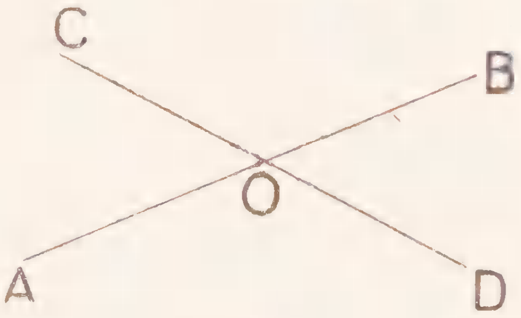

FIG. 29.

Measure with your protractor each pair of vertically opposite angles. Repeat this exercise with other intersecting straight lines. What do you find?

Suppose ∠AOC = 50°. How can you state, without measurement, the value of ∠BOC? Again, knowing the value of ∠BOC, what is the value of ∠BOD? This suggests a proof for the general case.

***THEOREM 4.—If two straight lines intersect, the vertically opposite angles are equal.**

Given two straight lines AB, CD intersecting at a point O.

To prove that the vertically opposite angles AOC, BOD are equal.

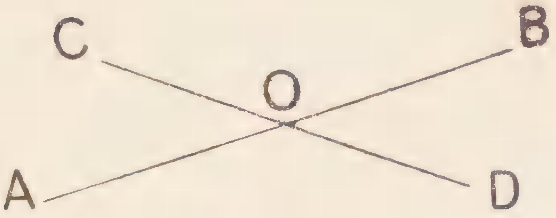

FIG. 30.

Proof. OC is a straight line standing on the straight line AB.

∴ ∠ AOC + ∠ BOC = 2 right ∠ s.

Again, OB is a straight line standing on the straight line CD.

∴ ∠ BOC + ∠ BOD = 2 right ∠ s.

∴ ∠ AOC + ∠ BOC = ∠ BOC + ∠ BOD.

∴ ∠ AOC = BOD.

In the same way it can be proved that the vertically opposite angles BOC, AOD are equal.

EXERCISE 7

1. In Fig. 30

(i) if ∠ AOC = 87°, calculate each of the angles COB, BOD, DOA.

(ii) if ∠ COB = 132°, calculate each of the angles BOD, DOA, AOC.

2. Draw two straight lines AB, CD intersecting at O such that ∠ BOD = 68°. What is the value of the ∠ s AOD, AOC?

Use your protractor to draw the bisector OE of ∠ BOD and the bisector OF of ∠ AOC. What is the sum of the ∠ s DOE, AOD, AOF? Are OE, OF in the same straight line and, if so, why?

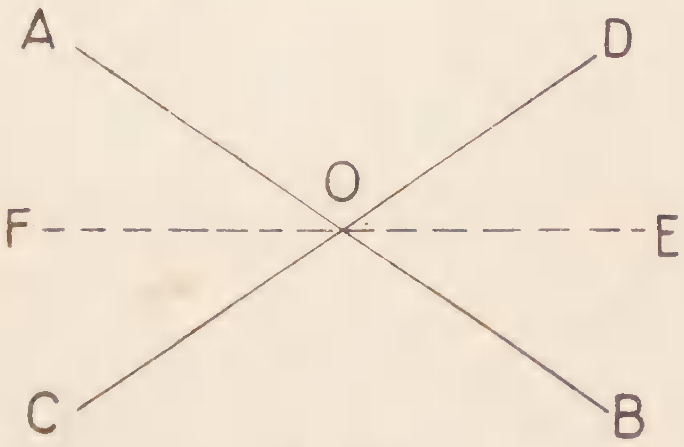

FIG. 31.

3. Two straight lines **PQRS** cut at a point **T**. If ∠ **PTR** = 44° what is each of the other angles?

4. Prove, by means of Theorems 3 and 4, that if **any** two straight lines intersect, the bisectors of a pair of vertically opposite angles are in the same straight line.

5. Give a proof of Theorem 4 by turning one of the straight lines, **AB** say, about the point **O**.

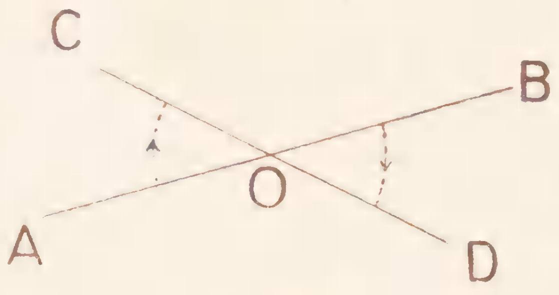

FIG. 32.

[If we turn the straight line **AB** about the point **O** then, when the part **OA** has moved to the position **OC**, it is clear that the part **OB** has moved to the position **OD**, for the line **AOB** remains straight. Hence ∠ **AOC** = ∠ **BOD**.]

CHAPTER III

PARALLELS

§ 1. When two straight lines such as AB, CD meet in a point, or can be made to meet in a point if produced far enough, they are said to have different directions. The angle at O, where they meet, measures the difference of direction of each straight line as compared with the other.

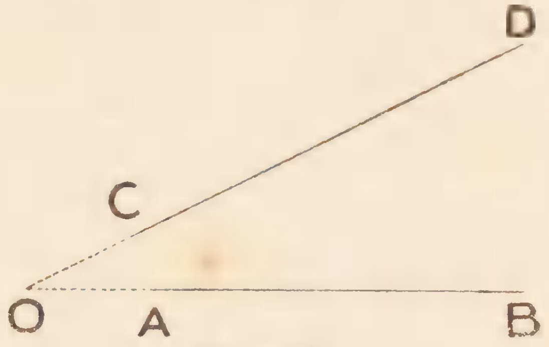

FIG 33.

Suppose there are two straight lines in a plane which will never meet however far they may be produced on either side. Lines ruled on a copy-book, the two edges of a ruler, railway lines on a straight track all suggest that such lines are possible. Straight lines of this kind may be said to have the **same direction.** For, if they could be made to meet, they would have different directions. Such lines are called **parallel** straight lines.

Hence we have the following definition :—

Def. Parallel straight lines are lines which are in the same direction and will therefore never meet however far they may be produced on either side.

Two straight lines drawn on a sheet of paper may **appear** to be parallel. Yet, if we could produce them far enough, they **might** meet at some distant point, a mile away for instance. It is obvious that we cannot use this method to test whether lines are parallel. Hence we must

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try to discover some other properties of parallel straight lines which will enable us to find whether two given straight lines are parallel or not.

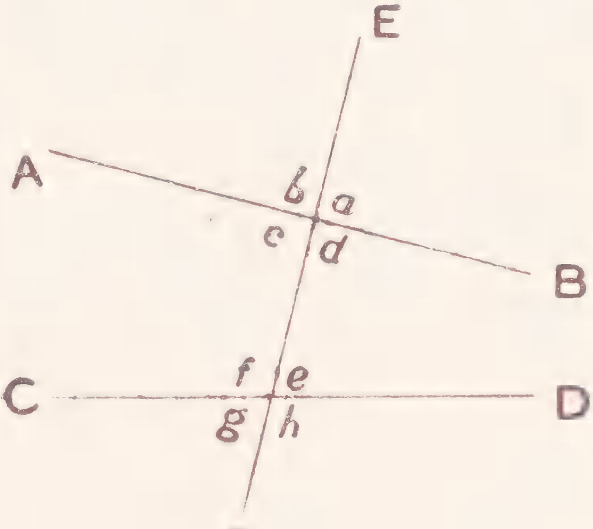

FIG. 34.

§ 2. Draw two straight lines AB, CD in **different** directions as in Fig. 34. Let any straight line EF cut them. Eight angles *a*, *b*, etc., are then formed.

The angles *a*, *b*, *g*, *h* which are outside the two lines AB, CD are called **exterior angles.**

The angles *c*, *d*, *e*, *f* which are inside the two lines are called **interior angles.**

The angles *a* and *e* which are on the same side of the cutting line EF, but one an exterior and the other an interior angle, are called a pair of **corresponding angles.** The other pairs are *b* and *f*, *g* and *c*, and *h* and *d*.

The angles *c* and *e* which are both interior angles but on opposite sides of the cutting line are called a pair of **alternate angles**. The other pair is *d* and *f*.

Measure these angles with your protractor. What is the least number of angles you need measure in order to obtain the value of each of the eight angles?

CORRESPONDING ANGLES

§ 3. Now use the two edges of your ruler to draw a pair of parallel straight lines. (As far as we can judge these two edges have the same direction and are therefore parallel.)

Measure each pair of corresponding angles. Do you find them to be equal? Test whether this is so for different positions of the cutting line. Are the corresponding angles equal in the previous exercise where the lines AB, CD are not parallel?

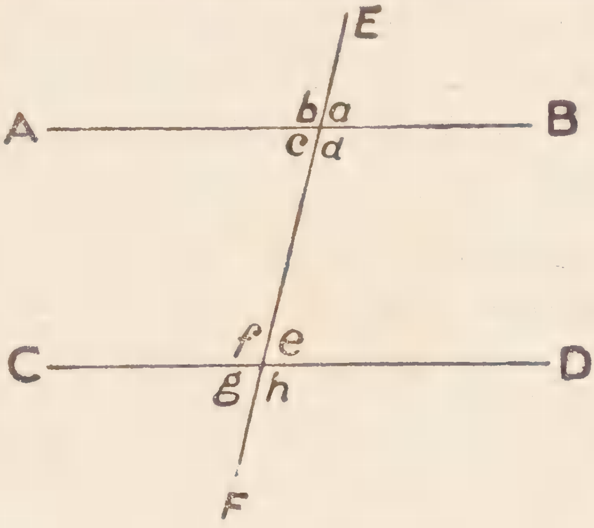

FIG. 35.

From these exercises we appear to have discovered something which distinguishes two parallel straight lines from two straight lines which are not parallel, namely, that in the former case the corresponding angles are equal, while in the latter they are not equal.

The following argument helps to verify what we have discovered. AB and CD, being parallel, have the same direction. It is therefore reasonable to suppose that the difference of direction of AB as compared with EF is equal to the difference of direction of CD as compared with EF. But the difference of direction of AB as compared with EF is measured by the angle a, and the difference of direction of CD as compared with EF is measured by the angle e. Therefore $\angle a = \angle e$.

Hence we are justified in assuming the truth of the following theorem.

***THEOREM 5.—If a straight line cuts two parallel straight lines, the corresponding angles are equal.**

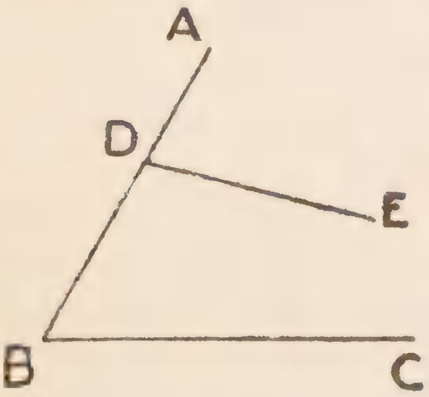

FIG. 36.

§ 4. Draw any angle ABC. Suppose, for example, it is 65°. At any point D in AB draw a straight line DE making an angle ADE, say 70°, not equal to the angle ABC. These two angles are corresponding angles but not equal. Are the straight lines DE and BC in the same direction?

Again draw any angle ABC and at any point D in AB draw, with the protractor, an angle ADE equal to the angle ABC. The two angles are now equal corresponding angles. In this case are DE and BC in the same direction?

From these exercises we are justified in assuming the truth of the following theorem.

***THEOREM 6.—If a straight line cuts two other straight lines so as to make a pair of corresponding angles equal, the two straight lines are parallel.**

Theorem 5 states that if two straight lines are parallel then the corresponding angles are equal. Theorem 6 is the other way about; it states that if the corresponding angles are equal then the lines are parallel. Such theorems are called **converse** theorems. We shall meet with further examples later on.

PRACTICAL CONSTRUCTION OF PARALLELS

§ 5. The fact stated in Theorem 6 provides us with a practical method of drawing through a given point a straight line parallel to a given straight line.

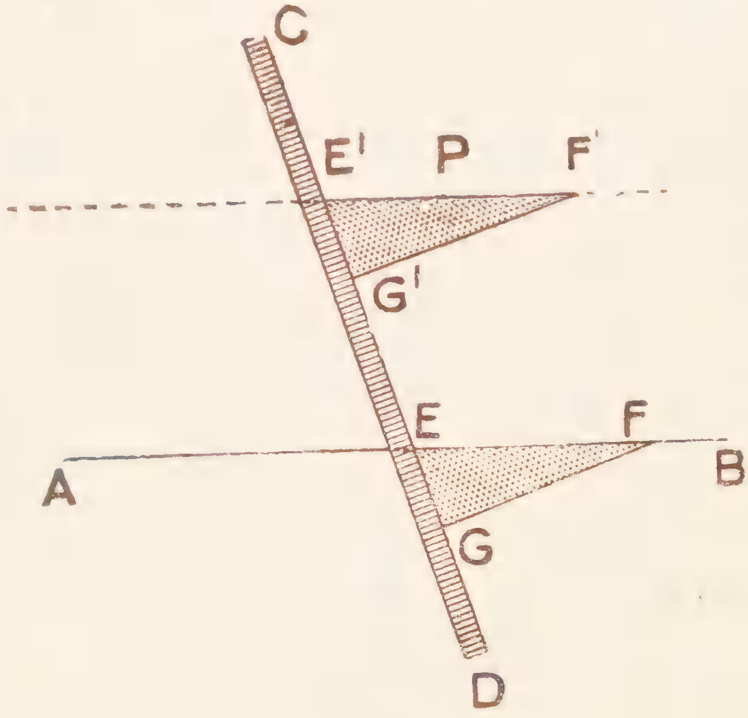

FIG. 37.

Let P be the given point and AB the given straight line. Place a set square EFG in such a way that the edge EF lies along the line AB.

Along the edge EG place a straight ruler CD. Holding the ruler firmly, slide the set square along the ruler until the edge **EF** passes just below the point **P** and now occupies the position **E′F′**. The straight line drawn through **P** along this edge is the straight line required. For, since the corresponding angles **GEF** and **G′E′F′** are equal, the straight lines **EF** and **E′F′** are parallel.

EXERCISE 8

1. A straight line **AB** cuts two parallel lines **PQ**, **RS** in the points **L**, **M**; write down the pairs of corresponding angles.

2. If in Fig. 35, ∠ $b = 110°$, find all the angles in the figure, giving your reasons.

3. If in Fig. 35, ∠ $e = 65°$, find all the other angles.

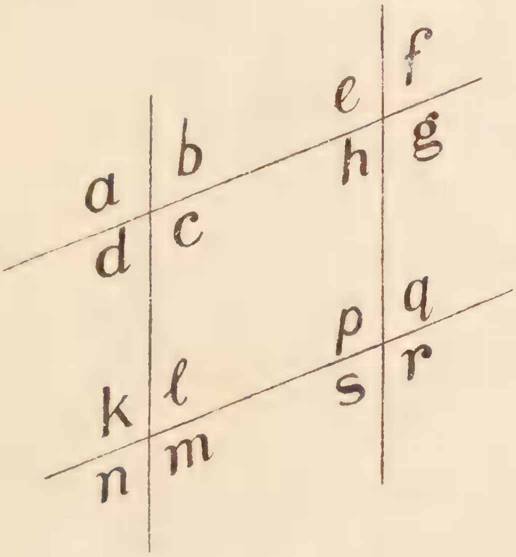

FIG. 38.

4. In Fig. 38, two parallel lines cut two other parallel lines forming the angles *a*, *b*, etc. If ∠ $k = 50°$, find all the other angles giving your reasons.

5. If in Fig. 38, ∠ $k = 120°$, find all the other angles.

6. In Fig. 39, **DE** is parallel to **AB**; and **DF** is parallel to **AC**; prove that ∠ **EDF** = ∠ **BAC**. [Produce **FD** to meet **AB**].

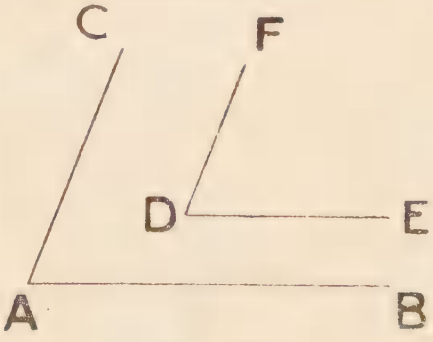

FIG. 39.

7. With ruler and set square draw a pair of parallel straight lines AB, CD. Take any point E in AB and draw a line EF making ∠ BEF = 66°. Produce FE to meet CD in G. With the aid of the protractor draw the bisectors EH, GK of the ∠s BEF and DGE. Are the straight lines EH, GK parallel?

[Is ∠ DGE = ∠ BEF? Why? Is ∠ EGK = ∠ FEH? Why? Are they corresponding angles?]

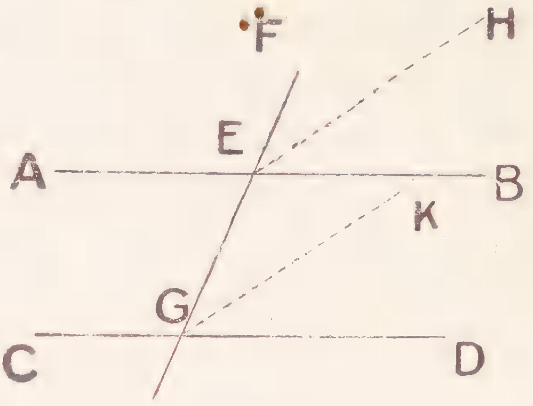

FIG. 40.

8. If a straight line meets two parallel straight lines prove that the bisectors of a pair of corresponding angles are parallel.

ALTERNATE ANGLES

§ 6. Using ruler and set square draw a pair of parallel straight lines. Draw a straight line cutting them as in Fig. 41.

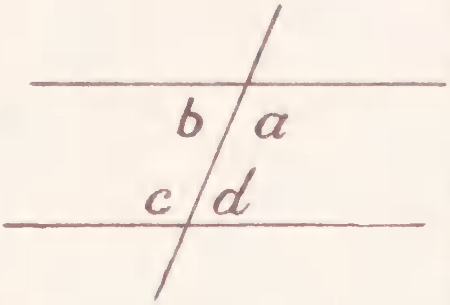

FIG. 41.

Measure a pair of alternate angles, *b* and *d* say. What do you observe? Repeat the exercise by taking different positions of the cutting line. Is the fact you have discovered true of the other pair of alternate angles *a* and *c*?

Draw any straight line AB standing on another straight line CD as in Fig. 42. Measure the angle ABD with the protractor and at A make an equal angle BAE. The angles ABD and BAE are alternate angles and are equal. Test with ruler and set square whether AE and CD are parallel.

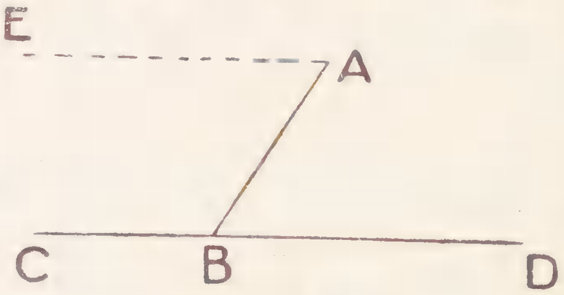

FIG. 42.

These exercises lead to the discovery of facts about alternate angles similar to those we already know about corresponding angles. We could, however, have found these out without making any measurements at all. If the statements made in Theorems 5 and 6 about corresponding angles are true, we can prove by a process of reasoning that similar statements about alternate angles are true also. In the course of the proof we make use of the fact stated in Theorem 4, that vertically opposite angles are equal.

*THEOREM 7.—**If a straight line cuts two parallel straight lines, the alternate angles are equal.**

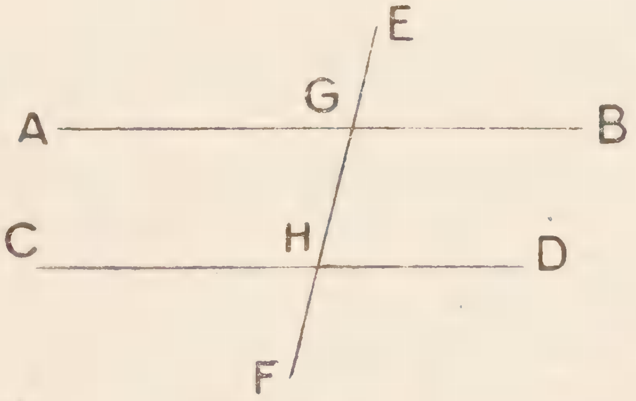

FIG. 43

Given the two parallel straight lines AB and CD, and a straight line EF cutting them at G and H.

To prove that ∠ AGH = the alternate ∠ GHD.

Proof. Because AB and CD are parallel and EF cuts them,

the ∠ EGB = the corresponding ∠ GHD.

But, ∠ EGB = the vertically opposite ∠ AGH.

∴ ∠ AGH = the alternate ∠ GHD.

Note.—It can be proved in the same way that the other pair of alternate angles, BGH and GHC, are equal. This also follows at once from the fact that ∠ BGH is the supplement of ∠ AGH, and ∠ GHC is the supplement of ∠ GHD.

***THEOREM 8.—If a straight line cuts two other straight lines so to make a pair of alternate angles equal the two straight lines are parallel.**

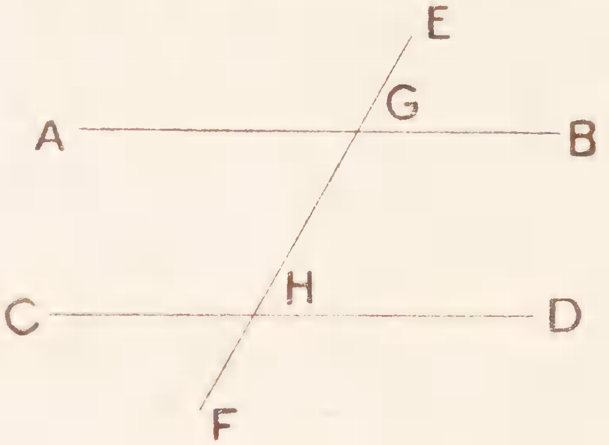

FIG. 44.

Given that the two straight lines AB, CD are cut by the straight line EF at G and H, and that the alternate angles AGH, GHD are equal.

To prove that AB and CD are parallel.

Proof. ∠ AGH = the vertically opposite ∠ EGB.

∴ ∠ EGB = ∠ GHD.

But these are corresponding angles.

∴ AB and CD are parallel.

Note.—Theorems 7 and 8 are converse theorems.

EXERCISE 9

1. If in Fig. 45, DAE is parallel to BC, ∠ BAC = 80° and ∠ EAC = 40°, find all the other angles, giving your reasons.

2. If in Fig. 45, ∠ DAB = 65°, ∠ BAC = 58° and ∠ ACB = 57°, prove that DE and BC are parallel.

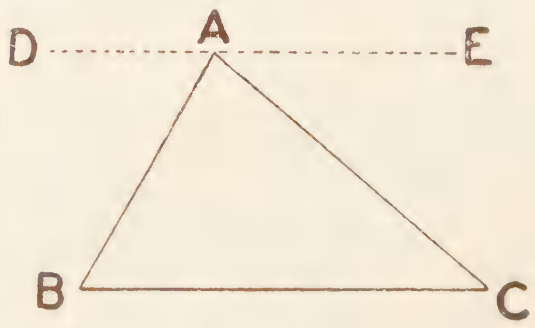

FIG. 45.

3. In Fig. 46, DE is parallel to FG; prove that $\angle c = \angle a + \angle b$. [Draw through H a line parallel to DE and FG.]

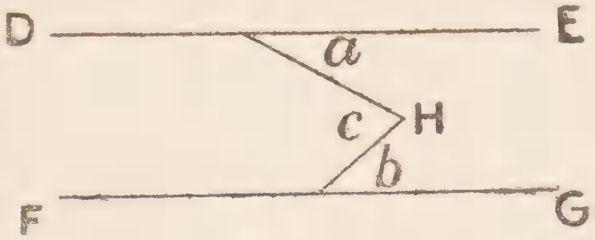

FIG. 46.

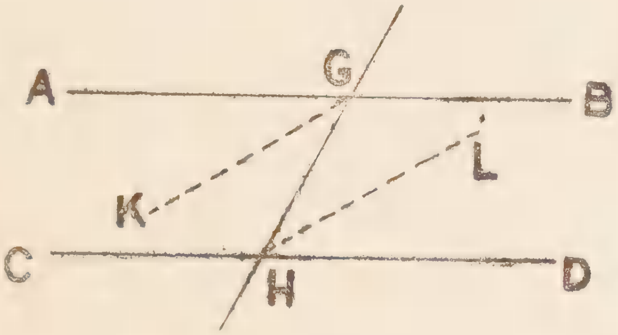

FIG. 47

4. In Fig. 47, AB and CD are parallel and KG, HL are the bisectors of the angles AGH, GHD. Prove that KG and HL are parallel.

INTERIOR ANGLES

§ 7. Draw a pair of parallel straight lines. Draw a straight line cutting them, forming the interior angles a, b, c, d. Measure a **pair of interior angles on the same side of the cutting line,** a and d say. What is their sum? What is the sum of the other pair b and c? Repeat the exercise with different figures. What fact is suggested?

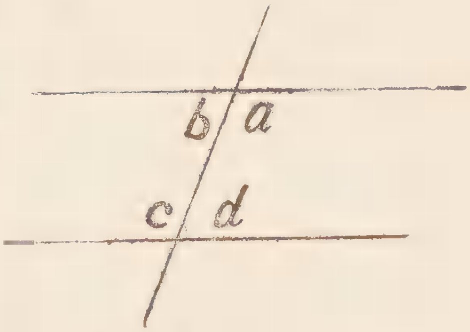

FIG. 48.

Draw any straight line AB standing on another straight line CD. Measure ∠ABD with your protractor. Suppose, for example, it is 74°. Draw at A an ∠BAE equal to the supplement of this angle, namely, 106°. Test with ruler and set square whether AE is parallel to CD.

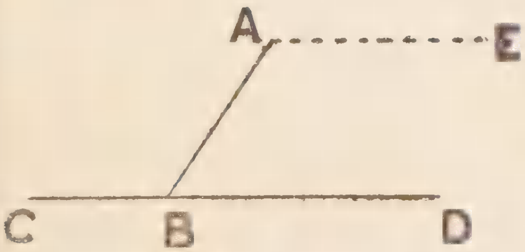

FIG. 49.

The above exercises lead to certain facts regarding parallels and interior angles on the same side of the cutting line. But, as in the case of alternate angles, the truth of those facts can be proved from what we already know about corresponding angles. In the course of the proof we make use also of the fact stated in Theorem 2.

***THEOREM 9.—If a straight line cuts two parallel straight lines, the sum of the interior angles on the same side of the cutting line is two right angles.**

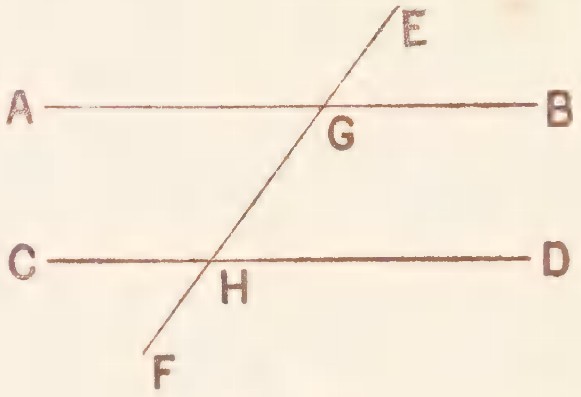

FIG. 50.

Given two parallel straight lines AB, CD and a straight line EF cutting them at G and H.

To prove that $\angle$ BGH + $\angle$ GHD = 2 right angles.

Proof. Because AB and CD are parallel, $\angle$ GHD = the corresponding $\angle$ EGB.

To each add the $\angle$ BGH.

Then $\angle$ BGH + $\angle$ GHD = $\angle$ BGH + $\angle$ EGB.
= a straight angle.

$\therefore$ $\angle$ BGH + $\angle$ GHD = 2 right angles.

*** THEOREM 10.—If a straight line cuts two other straight lines so that the sum of a pair of interior angles on the same side of the cutting line is two right angles, the two straight lines are parallel.**

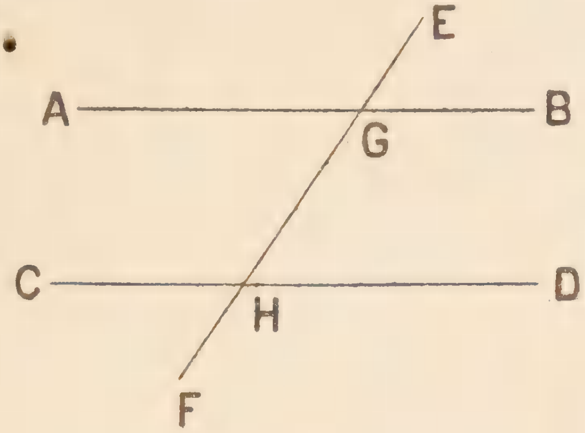

FIG. 51.

Given the straight line EF cutting the straight lines AB, CD so that

$\angle BGH + \angle GHD = 2$ right angles.

To prove that AB and CD are parallel.

Proof. $\angle EGB + \angle BGH =$ a straight angle.

$= 2$ right angles.

And, it is given that

$\angle BGH + \angle GHD = 2$ right angles.

$\therefore \angle EGB + \angle BGH = \angle BGH + \angle GHD.$

$\therefore \angle EGB = \angle GHD.$

And these are corresponding angles.

$\therefore$ AB and CD are parallel.

EXERCISE 10

1. In Fig. 52, AB is parallel to CD, and BC parallel to AD. If $\angle A = 60°$, find all the other angles of the figure, giving reasons.

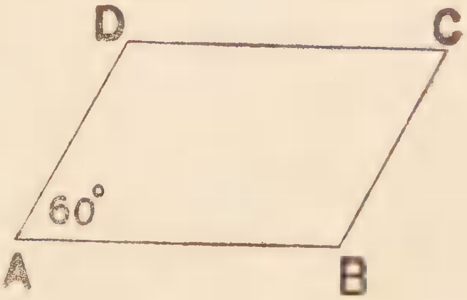

FIG. 52.

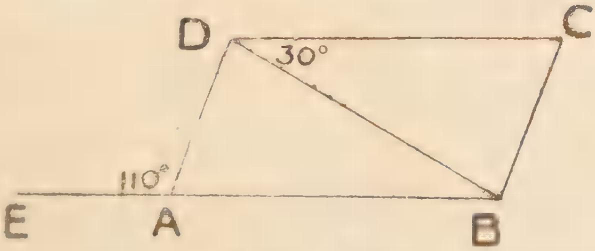

FIG. 53.

2. In Fig. 53, AB is parallel to CD, and BC parallel to AD, find all the angles of the figure.

3. Prove that if each of two straight lines is perpendicular to another straight line the two straight lines are parallel.

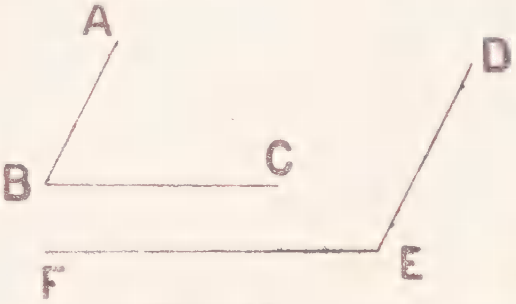

FIG. 54.

4. In Fig. 54, AB is parallel to DE and BC parallel to FE. Prove that the angles ABC, DEF are supplementary. [Produce BC to meet DE.]

§ 8. Suppose each of two straight lines AB, CD is parallel to a third straight line XY.

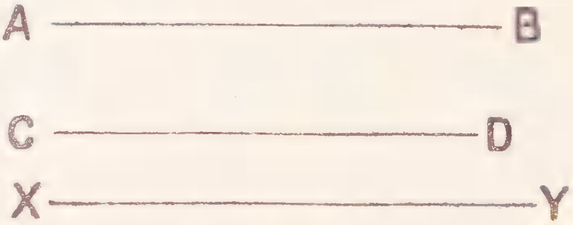

FIG. 55.

Then AB and CD each point in the same direction as XY. Therefore AB and CD point in the same direction as each other, that is, AB and CD are parallel.

The same result can be arrived at by making use of the facts stated in Theorems 5 and 6 concerning corresponding angles. The proof is as follows.

*** THEOREM 11.—Straight lines which are parallel to the same straight line are parallel to one another.**

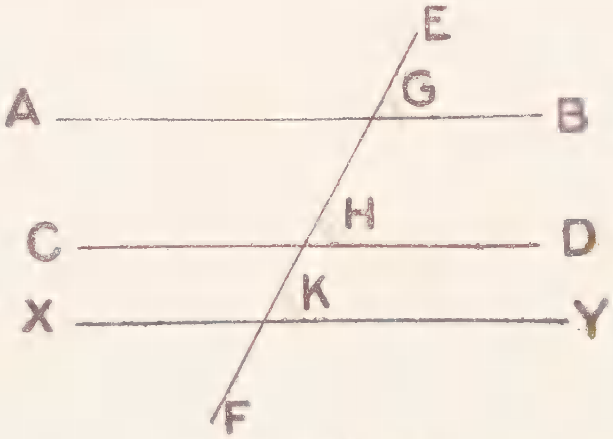

Fig. 56.

Given a straight line AB parallel to a straight line XY and also a straight line CD parallel to XY.

To prove that AB and CD are parallel.

Construction. Draw any straight line EF cutting AB, CD, XY in G, H, K respectively.

Proof. AB is parallel to XY.

∴ ∠EGB = the corresponding ∠GKY.

CD is parallel to XY.

∴ ∠GHD = the corresponding ∠HKY.

∴ ∠EGB = ∠GHD.

But these are corresponding angles.

∴ AB is parallel to CD.

SUMMARY OF RESULTS

The facts established in regard to parallels may be summarized as follows :—

When a straight line cuts two parallel straight lines,

(i) **Corresponding angles are equal.**

(ii) **Alternate angles are equal.**

(iii) **The interior angles on the same side of the cutting line are together equal to two right angles.**

Conversely, when a straight line cuts two other straight lines, if

(i) **A pair of corresponding angles are equal.**

or (ii) **A pair of alternate angles are equal.**

or (iii) **A pair of interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.**

Straight lines which are parallel to the same straight line are parallel to one another.

CHAPTER IV

DRAWING TO SCALE (EASY EXERCISES)

§ 1. Before a house or any other big building is constructed it is necessary first to have **plans** made of the various parts of the building. These plans are drawn to some convenient **scale.** Thus if the scale taken is 1 inch to 10 feet, what is meant is that an actual length of 10 feet in the building is represented on the plan by a length of 1 inch, and other lengths in proportion. A side of a room 34 feet long would be represented on this scale by a straight line 3·4 inches long.

Maps of countries are drawn to scale, the scale chosen depending on the size of map required. For example, in order to get a map of India within a square sheet of paper of side 6 inches it is necessary to take a scale of 1 inch to 400 miles. If, for example, we find by measurement that two towns marked on such a map are $1\frac{1}{2}$ inches apart then we know that the actual distance, in a straight line, between the two towns is roughly about 600 miles.

Example. A room is 38 ft. long and 23 ft. wide; draw a ground-plan of the room and find from it the distance between two opposite corners.

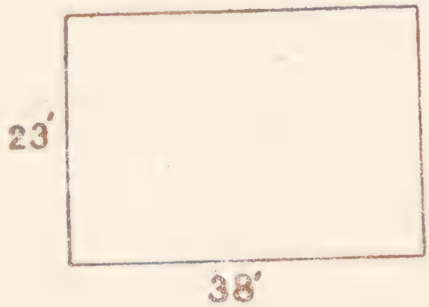

FIG. 57.

First make a freehand sketch of the required figure and mark on it the given lengths.

The next thing to do is to choose a convenient scale, the larger the better. If possible choose scales in which you have to divide or multiply by 10 or 100 only. Thus, in the given example, if a scale of 1 inch to 10 feet is taken, a length 38 ft. is represented by a line 3·8 in. long, and 23 ft. by 2·3 in. This gives a figure of convenient size.

Now draw the figure with these measurements, using the set square for making right angles. Write down the scale you are using underneath the figure.

If the figure has been drawn accurately you will find that the line joining one corner to the opposite corner is, to the nearest tenth of an inch, 4·4 in. long. Hence the actual distance between two opposite corners of the room is about 44 feet.

EXERCISE 11

1. What length in the scale-drawing represents
   (i) 54 ft. when the scale is 1 in. to 10 ft. ?
   (ii) 84 miles ,, ,, ,, 1 in. to 20 miles ?
   (iii) 6 miles 4 fur. ,, ,, ,, 2 in. to 1 mile ?
   (iv) 120 yds. ,, ,, ,, 1 cm. to 15 yds. ?

2. What actual length is represented by
   (i) 3·2 in. when the scale is 1 in. to 5 ft. ?
   (ii) 6·7 cm. ,, ,, ,, 1 cm. to 80 yds. ?
   (iii) 7·3 in. ,, ,, ,, 1 in. to 12 miles ?

3. In a map drawn to a scale of 1 in. to 400 miles find the approximate distance in miles between
   (i) Calcutta and Allahabad, the distance on the map being 1·3″.
   (ii) Lahore and Karachi ,, ,, ,, ,, ,, 1·6″.
   (iii) Rangoon and Mandalay ,, ,, ,, ,, ,, 0·95″.

4. If the scale is 1 in. to 150 miles, what length on the map represents the distance between
   (i) Madras and Calcutta, the actual distance being 840 miles ?
   (ii) Bombay and Nagpur ,, ,, ,, ,, 450 miles ?

5. Draw a plan of a door 7 ft. high and 4 ft. wide, taking a scale of ½ in. to 1 foot.

6. A ladder placed on level ground just reaches the top of a wall 18 ft. high. The foot of the ladder is 7 ft. away from the foot of the wall. Draw a plan (scale 1 in. to 5 feet) and find from it the length of the ladder.

7. A kite is attached to a string 150 ft. long, the other end of which is on the ground. Assuming that the string is straight and makes an angle of 58° with the ground, draw a figure to any convenient scale and find from it the height of the kite above the ground.

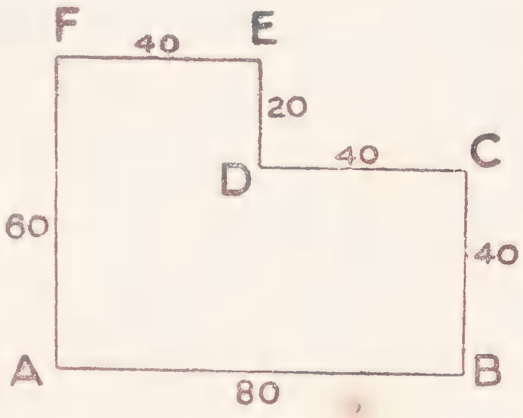

FIG. 58.

8. Fig. 58 is a rough sketch-plan of a piece of ground, of which the angles at the corners are right angles. The numbers denote the lengths of the sides in yards. Draw a plan (scale 1 in. to 20 yd.) and find from it the actual lengths of **AC**, **AD** and **AE**.

9. Along a straight level road **ABC** there are three upright posts **AP**, **BQ**, **CR** of which the heights are 25 ft., 30 ft. and 20 ft. respectively. **AB** = 40 ft. and **BC** = 30 ft. Draw a plan to any convenient scale and find from it the length of a tight wire **PQR** passing over the tops of the posts.

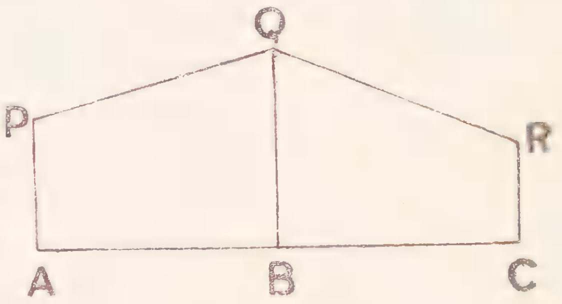

FIG. 59.

[Pupils should be given practice in measuring objects such as a blackboard, a window, the surface of a table, the floor of the class-room and draw plans to a scale chosen by themselves. They could also be given a map of the district in which they live and find from it the actual distances between various places.]

POINTS OF THE COMPASS

§ 2. The compass is an instrument which consists of a magnetic needle which can turn freely on a pivot about its middle point. This needle possesses the remarkable property of always pointing in the same direction. If you place the case which contains the needle on a table and turn the case round in any way, the needle will not turn with the case but will remain pointing in the same direction. This direction is called the **North** and **South** direction. The direction at right angles to this is the East and West direction.

Fig. 60 shows the points of the compass. **N** stands for North, **S** for South, **E** for East, and **W** for West.

The direction of the line which bisects the angle between North and East is called North-East (N.E.). S.E., S.W., N.W., have similar meanings.

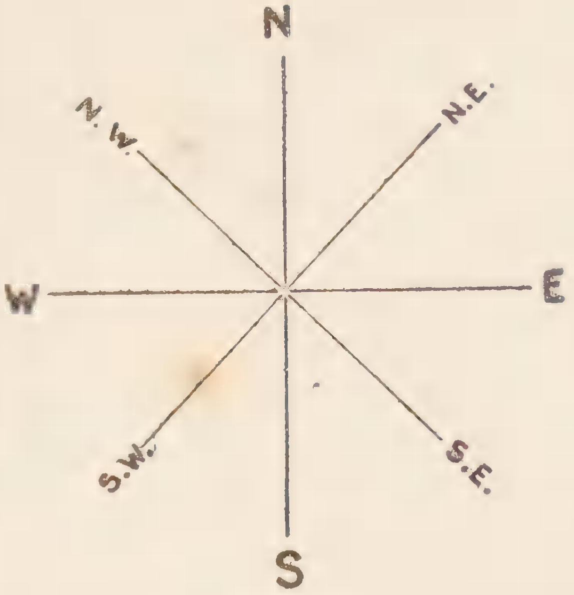

FIG. 60.

Suppose a person is standing at a point O and is facing North. If he makes a quarter-turn to the right, that is, turns through 90°, he is facing East. A second quarter-turn to the right makes him face South. A third quarter-turn to the right makes him face West, and finally a fourth quarter-turn to the right makes him face North again.

The compass enables mariners to steer a course at sea when out of sight of land.

On certain days of the year, namely, the 21st of March and the 23rd of September, the Sun rises in the East and sets in the West. On these two days the line joining the point on the horizon when the Sun rises to the point where it sets is the exact East and West line.

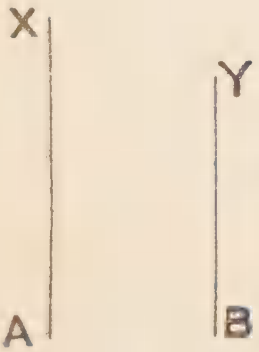

FIG. 61.

If a point X is north of a point A, and if a point Y is North of a point B, then AX and BY are parallel. Similarly for other directions given by the points of the compass.

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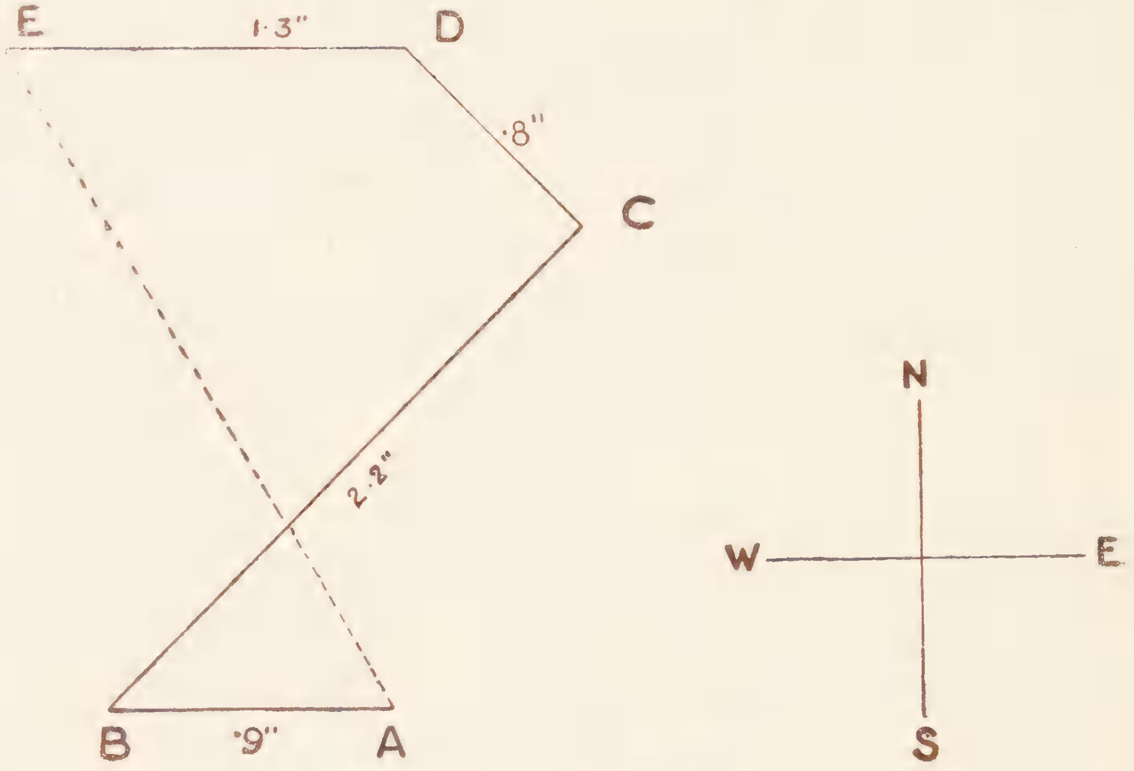
Example. A ship sails West for 9 miles from a certain port, then N.E. for 22 miles, then N.W. for 8 miles, and finally West again for 13 miles. Draw to scale a plan of her course and find by measurement the approximate distance of the ship from the port.



Scale $= \frac{1}{10}$ inch to 1 mile.
FIG. 62.

First draw a small figure showing the points of the compass. Take a scale of $\frac{1}{10}$ inch to the mile. This will give a figure of convenient size.

Let A be the position of the port. Then draw a line AB in the direction West making it ·9″ long. Draw BC 2·2″ long in the direction N.E., that is, making ∠ 45° with BA. Draw DC 8″ long in the direction N.W., that is, making ∠ 90° with CB. Finally draw DE 1·3″ long in the direction West, making ∠ 135° with DC.

Join AE. We find by measurement that AE is about 2·4″ long. Hence the approximate distance of the ship from the port from which she sailed is 24 miles.

EXERCISE 12

1. What is the angle between the following directions: S. and W., N. and N.W., W. and N.E., S.E. and S.W.?

2. Two persons start from the same place. One walks N.E. at 3 miles an hour and the other N.W. at 4 miles an hour. Draw a plan to show their positions at the end of four hours and find from it how far apart they are then.

3. A car travels 10 miles due West, then 5 miles due North, then 5 miles due East, then 10 miles due South. How far is the car from the starting point and in which direction ?

4. Indore is due W. of Calcutta at a distance of about 880 miles in a straight line. Mysore is due S. of Indore and S.W. of Calcutta. Draw a plan to a scale of 1 inch to 400 miles and find from it the approximate distance of Mysore from (1) Calcutta, (2) Indore.

5. Madras is S.E. of Bombay and S.W. of Puri at a distance of about 700 miles from each of the two towns. Draw a plan to a scale of 1 inch to 209 miles and find from it the approximate distance of Puri from Bombay.

6. Rangoon is S.E. of Calcutta at a distance of about 660 miles in a straight line and Mandalay is N. of Rangoon at a distance of 380 miles. Draw a plan to a scale of 1 inch to 200 miles and find from it the approximate distance of Mandalay from Calcutta.

Hence find how long it would take an aeroplane to fly from Mandalay to Calcutta at the rate of 100 miles an hour.

7. A person looking at the milestones on a straight road finds that he is N.E. of one and N.W. of another and that there are four others between these two milestones. How far is he from the road and from each of the six milestones ?

CHAPTER V

ANGLES OF A TRIANGLE

§ 1. A triangle is a plane figure bounded by three straight lines.

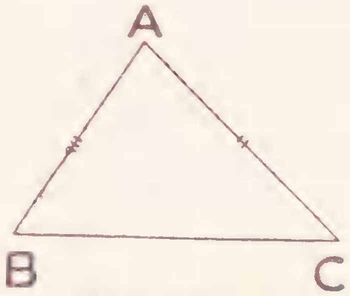
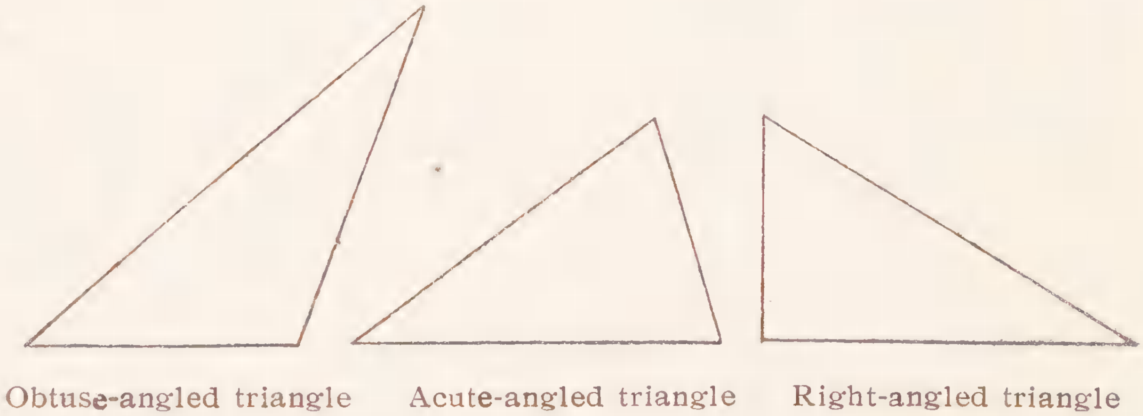


FIG. 63.

It has three sides and three angles at the three corners. A corner of a triangle is generally called a **vertex** or an **angular point.**

The abbreviation for triangle is △; thus the triangle ABC is written △ ABC.



Obtuse-angled triangle Acute-angled triangle Right-angled triangle

FIG. 64.

An **acute-angled triangle** has all its angles acute.

A **right-angled triangle** has one right angle.

An **obtuse-angled triangle** has one obtuse angle.

In a right-angled triangle the side opposite the right angle is called the **hypotenuse.**

§ 2. Draw (*a*) any acute-angled triangle, (*b*) any right-angled triangle, (*c*) any obtuse-angled triangle. Find the sum of the angles of each triangle, measuring the angles with your protractor.

Repeat this exercise with triangles of different shapes. What fact do you discover regarding the sum of the angles of a triangle?

§ 3. Draw any three straight lines cutting each other, as in Fig. 65, and forming a △ ABC. Place a straight pencil along BC and pointing in the direction BC. Turn the pencil clockwise about the point C through ∠ BCA; it now lies along the line CA.

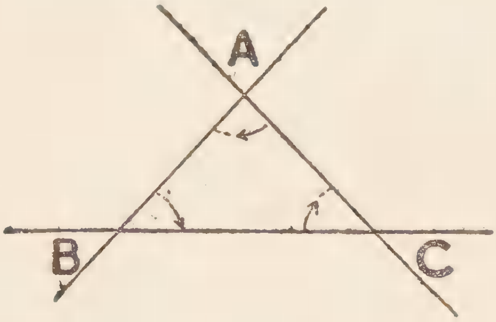


FIG. 65.

Again, turn the pencil clockwise about the point A through ∠ CAB; it now lies along the line AB. Finally, turn the pencil clockwise about the point B through ∠ ABC. The pencil has now returned to its original line BC but is pointing in the direction CB, that is, opposite to its original direction. The pencil has therefore made half a complete turn. What then is the sum of the angles BCA, CAB and BAC?

§ 4. From the above exercises we infer that the sum of the angles of a triangle is two right angles. We can, however, **prove** that this is true by making use of the facts we already know concerning parallels.

THEOREM 12.—The sum of the angles of a triangle is equal to two right angles.

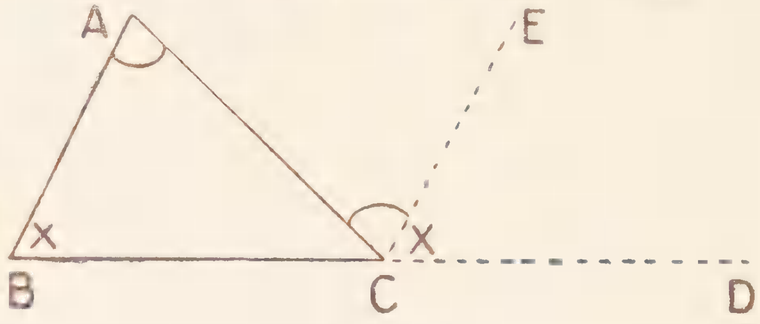


FIG. 66.

Given a triangle ABC.

To prove that ∠ BAC + ∠ ABC + ∠ ACB = 2 right angles.

Construction. Produce BC to D.
Draw CE parallel to BA.

Proof. AB and CE are parallel and AC cuts them.
∴ ∠ BAC = the alternate ∠ ACE.
AB and CE are parallel and BC cuts them.
∴ ∠ ABC = the corresponding ∠ ECD.
∴ ∠ BAC + ∠ ABC = ∠ ACE + ∠ ECD
= ∠ ACD.
To each side of the equation add ∠ ACB.
∴ ∠ BAC + ∠ ABC + ∠ ACB = ∠ ACD + ∠ ACB
= 2 right angles.

for BCD is a straight line.

∴ the sum of the angles of the △ ABC is equal to 2 right angles.

COROLLARIES TO THEOREM 12

§ 5. A fact which follows easily from the proof of a theorem is called a **corollary.**

Cor. 1. If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

Thus in Fig. 66, the exterior $\angle$ ACD is equal to the sum of the two interior opposite angles BAC and ABC.

This is proved in the course of the proof of Theorem 12.

Cor. 2. If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other.

Cor. 3. If one angle of a triangle is a right angle, the sum of the remaining two angles is a right angle.

EXERCISE 13

1. In a triangle ABC, $\angle B = 75^\circ$, $\angle C = 39^\circ$; calculate the $\angle A$. Draw such a triangle and check your result by measurement.

2. In a triangle XYZ, $\angle Y = 107^\circ$, $\angle Z = 44^\circ$; calculate the $\angle X$. Draw such a triangle and check your results by measurement.

3. Is it possible to have triangles with angles of
   (i) 80°, 80°, 20°. (ii) 69°, 71°, 39°. (iii) 23°, 125°, 33°.
   (iv) 100°, 48°, 32°. (v) 35°, 36°, 110°. (vi) 57°, 58°, 65°?

4. What is the third angle of a triangle when two of the angles are—
   (i) 87° and 34°. (ii) 9° and 18°. (iii) 116° and 49°.
   ✓ (iv) 54° and 72°. (v) 91° and 87°. (vi) 36° and 128°?

5. If the three angles of a triangle are equal, how many degrees are there in each angle?

✓6. The side AB of a triangle ABC is produced to D. If $\angle CBD = 128^\circ$ and $\angle A = 84^\circ$, calculate the $\angle C$.
Construct such a triangle and check your result by measurement.

✓

7. In Fig. 67, if $\angle BAD = 41^\circ$, $\angle AEF = 78^\circ$, $\angle BDE = 66^\circ$, and $\angle ACB = 52^\circ$, calculate the values of the angles ABE, EBD, EFC, EAF.

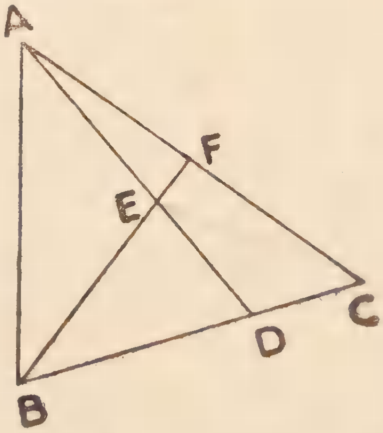


FIG. 67.

8. ABC is a triangle; BC is produced to D and CB to E. If ∠ ACD = 79° and ∠ ABE = 154°, calculate ∠ A.

Draw such a triangle and check your result by measurement.

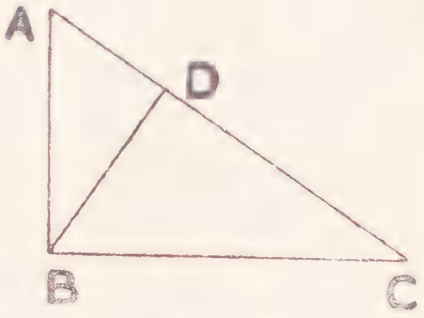


FIG. 68.

9. In Fig. 68, ABC is a right-angled triangle with the right-angle at B. BD is drawn perpendicular to AC.

If ∠ C = 33°, find the values of the remaining angles in the figure.

10. Prove that the sum of the angles of a triangle ABC is equal to two right angles, by drawing through A a line DE parallel to BC.

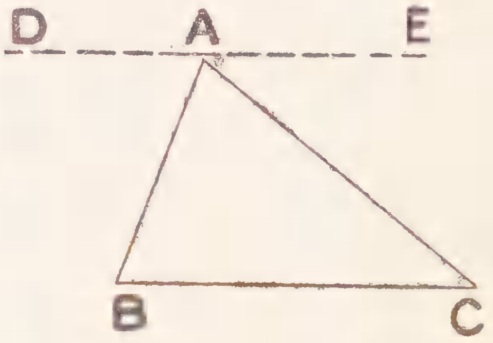


FIG. 69.

ANGLES OF A POLYGON

§ 6. A **polygon** is a plane figure bounded by straight lines.

The sum of the sides of a polygon is called its **perimeter.**

A polygon in which each interior angle is less than a straight angle is said to be **convex.** The polygon in Fig. 70 is not convex, because the interior angle at C is greater than a straight angle.

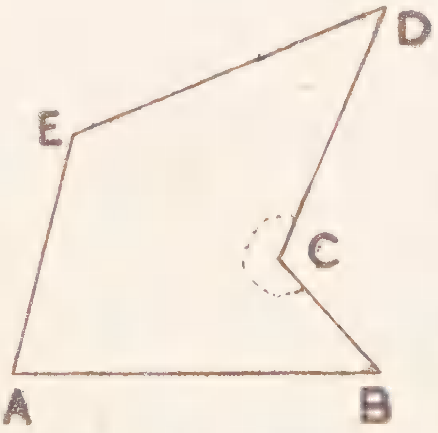


FIG. 70.

[Unless otherwise stated, all polygons referred to in this book are to be taken as convex polygons.]

§ 7. A polygon which has four sides is called a **quadrilateral.**

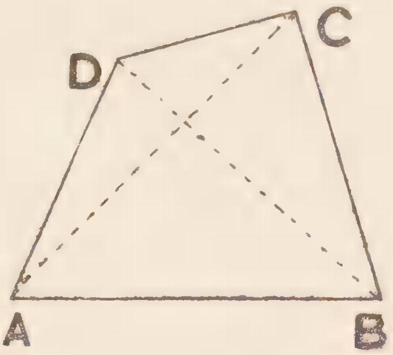


FIG. 71.

The straight lines joining opposite vertices (corners) of a quadrilateral are called **diagonals.**

Thus in Fig. 71, AC, BD are diagonals of the quadrilateral ABCD.

A diagonal divides a quadrilateral into two triangles, and the sum of the angles of the quadrilateral is equal to the sum of the angles of the two triangles.

Hence the sum of the angles of a quadrilateral is 4 right angles.

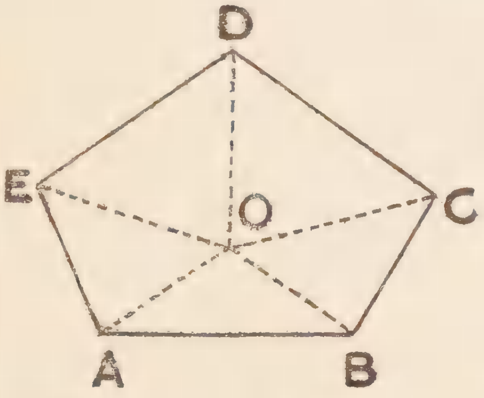


FIG. 72.

§ 8. Consider Fig. 72. It is a polygon of 5 sides and is called a pentagon. Take any point O within the pentagon and join it to the vertices. You then get 5 triangles, the same as the number of sides.

The sum of the angles of these triangles is 10 right angles.

Hence the sum of the angles of the pentagon together with the sum of the angles at O is 10 right angles.

But the sum of the angles at O is 4 right angles.

Therefore, the sum of the angles of the pentagon is 6 right angles.

Note that in naming a polygon the letters must be taken *in order* round the figure. It does not matter at which point you start or in which direction you go round, provided only that the letters are taken in order. Thus the above pentagon may be called ABCDE or CDEAB or EDCBA, but not ACDEB or EBACD.

§ 9. Repeat the exercise of § 8 with polygons of 6, 7, 8 and 10 sides, dividing each polygon into triangles by joining the vertices to any point within the polygon. Tabulate your results as follows :—

Number of sides of polygon.	Number of triangles	Sum of all the angles	Sum of all the angles at O.	Sum of angles of polygon.
5 (pentagon)				
6 (hexagon)				
7 (heptagon)				
8 (octagon)				
10 (decagon)				

The same method enables us to find a formula for the sum of the angles of a polygon of n sides, where n stands for any number.

THEOREM 13.—The sum of the angles of a polygon of n sides is $2n-4$ right angles.

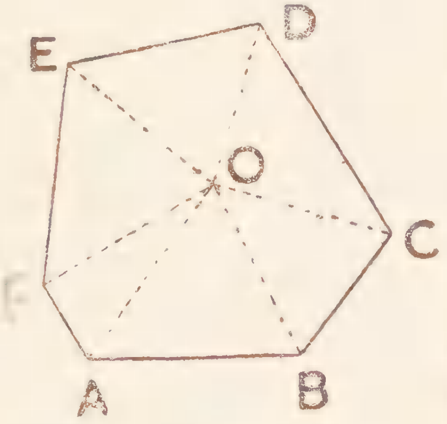


FIG. 73.

Given a polygon ABCD . . . of n sides.

To prove that $\angle ABC + \angle BCD + \angle CDE +$ etc. $= 2n - 4$ right angles.

Construction. Join the vertices A, B, C, . . . to any point O inside the polygon.

There are then n triangles OAB, OBC, etc.

The sum of the angles of each triangle is 2 right angles.

Therefore the sum of the angles of the n triangles is $2n$ right angles.

But the sum of the angles at O is 4 right angles.

Hence the sum of the remaining angles ABC, BCD, etc. is $2n-4$ right angles.

Cor. If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is 4 right angles.

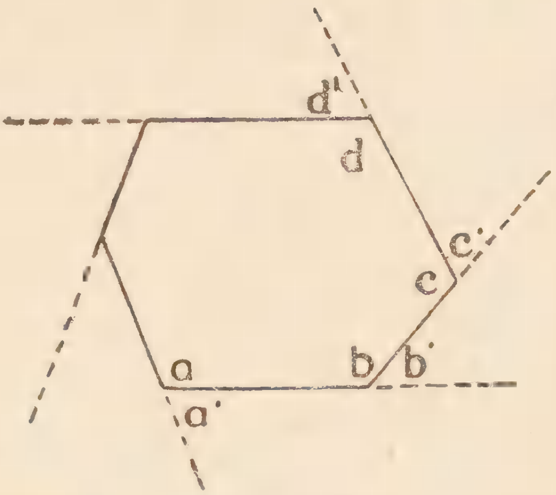


FIG. 74.

Suppose the polygon has n sides. Denote the interior angles by a, b, etc., and the corresponding exterior angles by a', b', etc.

We then have

$a + a' = 2$ right angles.
$b + b' = 2$ right angles,
etc.

Since the polygon has n sides there are n pairs of angles. Hence, by addition,

$(a + b + c +$ etc.$) + (a' + b' + c' +$ etc.$) = 2n$ right angles or, **a** + **b** + **c** + etc. $= 2n$ right angles $- (a' + b' + c' +$ etc.$)$

But $a + b + c +$ etc. $= 2n - 4$ right angles.

$\therefore a' + b' + c' +$ etc. $= 4$ right angles.

§ 10. A polygon is said to be **regular** when all its sides are equal and all its angles are equal.

It has been shown that the sum of the angles of a quadrilateral is 4 right angles. Hence in a regular quadrilateral each angle is a right angle; such a quadrilateral is called a **square.** Thus a square has 4 equal sides and each of its angles is a right angle.

To calculate the number of degrees in an angle of a regular polygon we can use either Theorem 13 or its corollary. For example, to find the number of degrees in an angle of a regular polygon of 9 sides (9-gon.).

Method I. The sum of the angles.

$$= (9 \times 2) - 4 \text{ right angles.}$$
$$= 14 \text{ right angles.}$$

Since the polygon has 9 angles, which are all equal

$$\text{each angle} = \frac{14}{9} \text{ of a right angle.}$$
$$= \frac{14}{9} \times 90^\circ.$$
$$= 140^\circ.$$

Method II. The sum of the 9 exterior angles is 4 right angles.

$$\therefore \text{ each exterior angle} = \frac{4}{9} \text{ of a right angle.}$$
$$= 40^\circ.$$
$$\therefore \text{ each interior angle} = 180^\circ - 40^\circ.$$
$$= 140^\circ.$$

CONSTRUCTION OF REGULAR POLYGONS

When the angle of a given regular polygon has been calculated, such a polygon can easily be constructed by means of the protractor.

Suppose it is required to draw a regular polygon of 9 sides, each side being 2″ long. First draw a straight line AB 2″ long. With the protractor draw ∠ ABC = 140°, making BC = 2″. Then draw ∠ BCD = 140°, making CD = 2″. And so on. If the measurements have been made accurately the end of the last line should fall on the first point A.

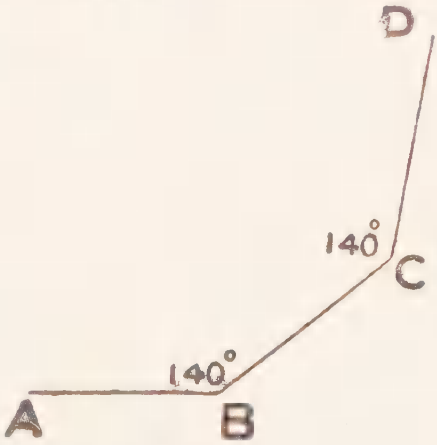


FIG. 75.

EXERCISE 14

1. Find the sum, in right angles, of the interior angles of a polygon of (i) 12, (ii) 15, (iii) 20 sides.

2. If three angles of a quadrilateral are 156°, 49° and 74°, what is the fourth angle?

3. On a straight line 6·8 cm. long construct a square. Find the length of each diagonal.

4. Find the number of degrees in an angle of a regular pentagon. Construct a regular pentagon having each side 2·5″ long.

5. Find the number of degrees in an angle of a regular hexagon. Construct a regular hexagon having each side 5·4 cm. long.

6. Find the number of degrees in an angle of a regular octagon. Construct a regular octagon having each side 2″ long.

7. Find the number of degrees in an angle of a regular polygon of (i) 16, (ii) 18, (iii) 30 sides.

8. Draw any pentagon **ABCDE**. Join any one vertex, **A** say, to each of the other vertices. How many triangles are formed? By how much is the number of triangles formed less than the number of sides of the pentagon? Hence calculate the sum of the angles of the pentagon.

9. Use the method of Ex. 8 to find the sum of the angles of polygons of 6, 7, 8 and 10 sides and compare the results with your results in § 9.

10. Prove Theorem 13 by the method of Ex. 8.

NOTE ON THE NATURE OF A GEOMETRICAL PROOF

We are now in a position to consider what exactly is meant by a geometrical proof.

In a court of law it often happens that a person is 'proved' to be guilty of a certain offence, theft say, though nobody has actually seen him do it. Various circumstances, such as being seen loitering about the house from which the things were stolen, or being found with the stolen things in his possession, may point to his guilt. But, strictly speaking, this does not 'prove' that the accused person is guilty; all that can be said for certain is that it is extremely likely that it is he who committed the theft. Or, as lawyers would say, the 'circumstantial evidence' against him is very strong.

In the same way no geometrical fact can be 'proved' by measurements alone. For example, by drawing a number of triangles and measuring their angles, our measurements point to the fact that the sum of the angles of a triangle is 180°. But, however many triangles we may test in this way, we cannot be absolutely certain that the result is true for all triangles, although it is extremely likely that such is the case. In other words, the 'circumstantial evidence' in favour of the result being true for all triangles is very strong.

The proof of a theorem is of a different nature. In the first place we are **given** certain facts. Thus, in Theorem 5 we are given a straight line cutting two parallel straight lines; in Theorem 12 we are given simply a triangle. What is given is called the **data.** Another name for it is **hypothesis,** which means 'what is supposed.' From the given facts we arrive at other facts by a process of reasoning only. The fact arrived at, or proved, is called the **conclusion.** Thus, in Theorem 12 the conclusion is that the sum of the angles of a triangle is two right angles.

In proving a theorem we are allowed to make use of definitions, axioms, and facts which have already been proved in previous theorems. Thus, in Theorem 12, before we can prove anything about a triangle there must be no confusion as to what is meant by a triangle, that is to say, we require beforehand a definition of a triangle. An axiom, as has already been mentioned, is a statement the truth of which is obvious. For example, 'things which are equal to the same thing are equal to one another' is an axiom. Another example is :—'if equals be added to equals the wholes are equal,' or, putting it in the form of an equation, 'if $a = b$, then $a + c = b + c$.' This particular axiom is used in the proof of Theorem 12.

The proof of some theorems depends on definitions only. Thus the geometrical fact stated in Theorem 1 follows at once from the definition of a right angle. The proof of other theorems depends on facts proved in previous theorems. Thus Theorem 4 (vertically opposite angles) is proved by means of Theorem 2 and the axiom 'if $a = b$, then $a - c = b - c$.' Theorem 7 (alternate angles) is proved by means of Theorem 5 (corresponding angles) and Theorem 4. Theorem 12 is proved by means of Theorems 5 and 7 and the axiom 'if $a = b$, then $a + c = b + c$.' It will be seen later that some theorems depend on three or more previous theorems for their proof.

A theoretical proof is an absolute certainty **provided that** we base our reasoning only on (i) definitions which have already been stated, (ii) axioms and (iii) facts already proved in previous theorems. For example, in the proof of Theorem 12, if the statements made regarding alternate and corresponding angles are true, then the conclusion regarding the sum of the angles of a triangle is true. If these statements are not true, then the proof falls to the ground.

We can now see how a sequence, or chain, of theorems is built up. It is important to bear in mind that a theorem cannot be proved by means of a theorem which comes **after** it in the sequence. Such an attempt is like trying to build the upper part of a house before the lower part has been erected.

CHAPTER VI

CONSTRUCTION OF TRIANGLES

§ 1. Suppose, for example, it is required to construct a triangle ABC, given that $BC = 3''$ and $\angle B = 64^\circ$.

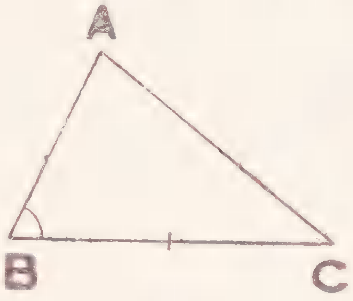


FIG. 76.

It is advisable first to make a freehand sketch of the triangle and mark on it the given parts. This helps us to see what line or angle should be drawn first.

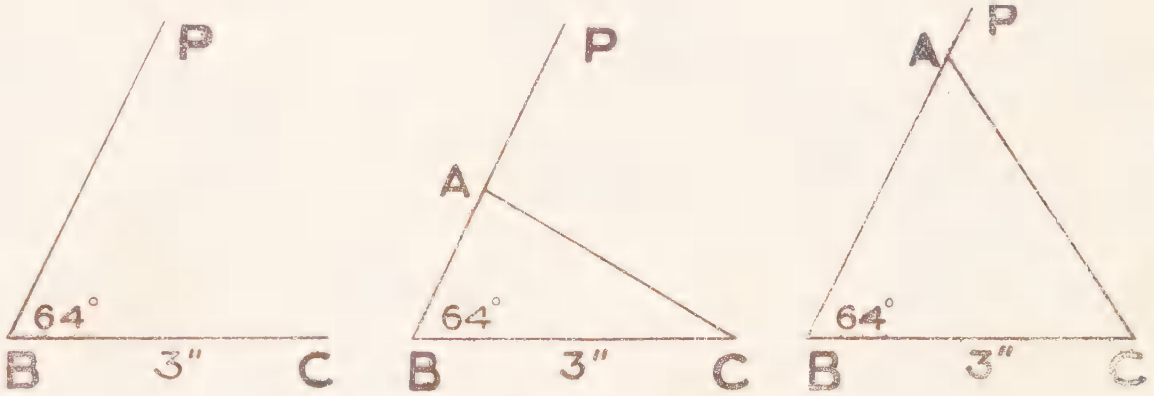


FIG. 77.

In this particular case the obvious thing to do is first to draw a straight line $BC = 3''$. Then, using the protractor, draw at B a straight line BP making $\angle CBP = 64^\circ$. It is clear that the point A may be anywhere on the straight line BP, for any such triangle contains the given parts, namely, $BC = 3''$ and $\angle B = 64^\circ$ (Fig. 77).

Hence in this case we can construct any number of triangles **of different sizes and shapes,** each one of which contains the given parts.

Draw a triangle ABC, given

(1) $AB = 3''$, $AC = 2{\cdot}3''$.
(2) $BC = 2{\cdot}8''$, $\angle A = 47^\circ$.
(3) $\angle B = 82^\circ$, $\angle C = 39^\circ$.

In each case it will be found that any number of triangles of different sizes and shapes can be drawn to satisfy the given requirements.

§ 2. A triangle has 6 parts—the 3 sides and the 3 angles. We have seen from the above exercises that when only two of these parts are given, any number of triangles of different sizes and shapes can be drawn to contain those parts.

What then is the least number of parts that must be given in order that only one kind of triangle may be possible, that is, a triangle of one definite size and shape? It will be seen from the constructions which follow that, in general, **three** parts are sufficient.

§ 3. **Case I. To construct a triangle, given two sides and the included angle.**

Suppose, for example, it is required to draw a triangle ABC with the side AB = 2·5″, AC = 3″, and the included ∠ BAC = 43°.

Draw first a freehand sketch of a triangle ABC and mark on it the given parts. You can then proceed with the proper construction.

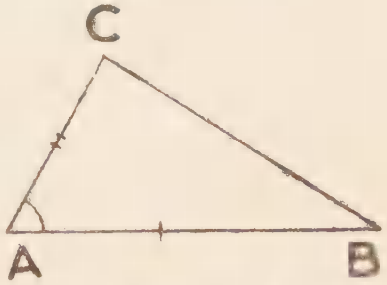


FIG. 78.

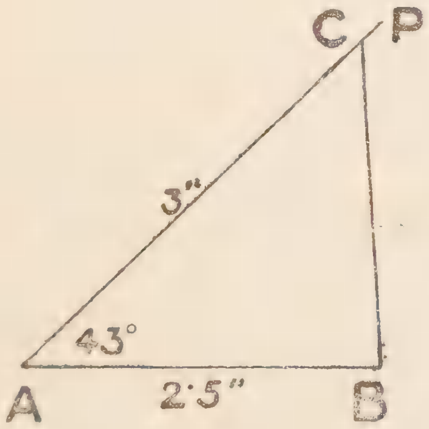


FIG. 79.

Draw one of the given lines, AB say, making it of the required length 2·5″. Then at A, with the protractor draw a line AP making ∠ BAP = 43°. Cut off from AP a length AC = 3″. Join BC. Then ABC is a triangle which contains the given parts.

EXERCISE 15

1. Draw a △ ABC to the following measurements, and in each case measure the remaining side and angles :—

(i) BC = 3·2″, AB = 1·8″, ∠ B = 58°.
(ii) ∠ C = 74°, BC = 6·8 cm., CA = 5·2 cm.
(iii) AB = 7·6 cm., AC = 3·2 cm., ∠ A = 107°.

2. Draw a △ XYZ to the following measurements and in each case measure the remaining side and angles :—

(i) XY = 2·7″, ∠ X = 28°, XZ = 4″.
(ii) ∠ Z = 135°, YZ = 4·5 cm., ZX = 4·5 cm.
(iii) ∠ Y = 60°, YZ = 2·9″, XY = 3·4″.

3. Draw two △'s ABC, DEF such that AB = DE = 3·3″, AC = DF = 4·2″, and ∠ A = ∠ D = 50°.

Measure the remaining side and angles of each △ and compare the results. Are these two △'s of the same size and shape ?

4. Draw two △'s LMN, PQR such that ∠ M = ∠ Q = 122°. LM = PQ = 5·4 cm., MN = QR = 7·2 cm.

Measure the remaining side and angles of each △ and compare the results. Are these two △'s of the same size and shape ?

§ 4. From the above exercises we find that, when we have to draw a triangle two of whose sides and included angle are given, the resulting triangle is always of the same size and shape. In all cases we find that the remaining side and angles have the same measurements.

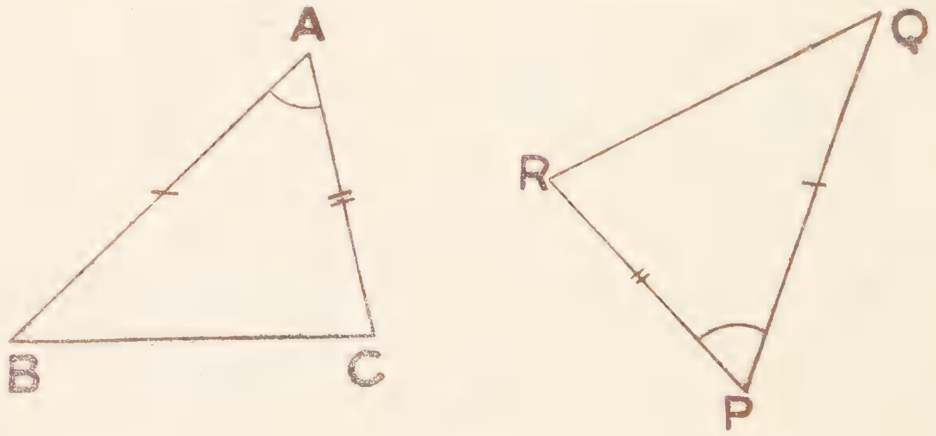


FIG. 80.

Thus, in Fig. 80, the triangles ABC, PQR differ in position but, if AB = PQ, AC = PR and ∠ A = ∠ P, we find by measurement that the remaining side BC of the triangle ABC is equal to the remaining side QR of the triangle PQR, and the remaining angles B, C of the triangle ABC are respectively equal to the remaining angles Q, R of the triangle PQR.

The angles B and Q are said to **correspond** to each other because they are opposite the two given equal sides AC, PR. Similarly the angles C and R are corresponding angles, being opposite the given equal sides AB, PQ. Also BC, QR are corresponding sides being opposite the given equal angles A, P.

When two triangles have the three sides of the one respectively equal to the three sides of the other, and the three angles of the one equal to the corresponding three angles of the other, they are said to be **equal in all respects** or **congruent.**

From the results of Exercise 15 we are justified in assuming that the following statement is true :—

If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, then the two triangles are congruent.

The theoretical proof of this statement is contained in Theorem 14.

§ 5. **Case II. To construct a triangle, given two angles and a side.**

(*a*) Let the two given angles be adjacent to the given side.

Suppose, for example, it is required to draw a triangle ABC with ∠ B = 74°, ∠ C = 42°, and BC = 2·8″.

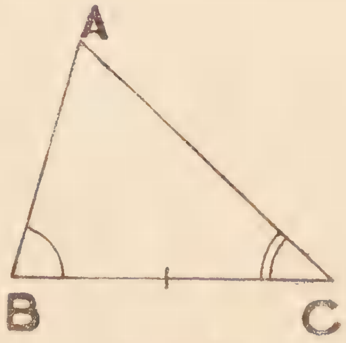


FIG. 81.

As in Case I, make first a freehand sketch of the triangle and mark on it the given parts.

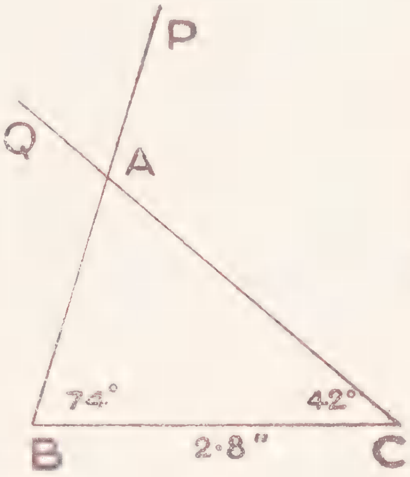


FIG. 82.

Draw a straight line BC 2·8″ long and with your protractor draw ∠CBP = 74° and ∠BCQ = 42°. Let BP, CQ meet at A. Then ABC is the required triangle.

(*b*) Let only one of the two given angles be adjacent to the given side.

Suppose, for example, it is required to draw a triangle ABC with ∠A = 72°, ∠B = 57° and BC = 6 cm. As before begin by drawing a straight line BC = 6 cm., then with the protractor make ∠CBP = 57°. We know that the point A is to be somewhere on the line BP. How are we to find its position? We are not allowed to do it by guess-work. The position of A can be found by making use of the fact proved in Theorem 12, namely, that the sum of the angles of a triangle is 180°. Hence, ∠C = 180° − 72° − 57° = 51°. All we have to do therefore is to draw at C an ∠BCQ = 51°. Where BP and CQ meet is the required point A.

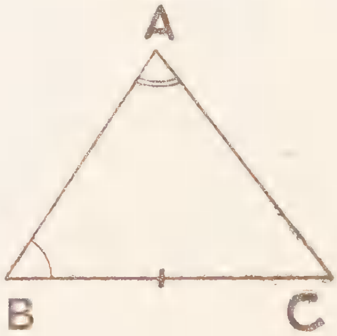


FIG. 83.

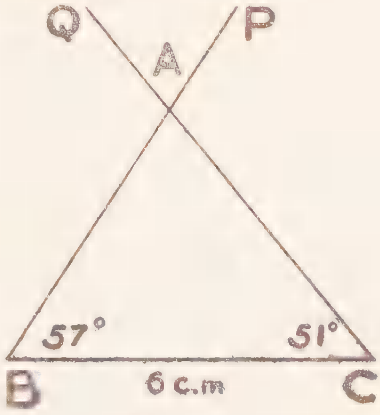


FIG. 84.

EXERCISE 16

1. Draw a $\triangle ABC$ to the following measurements, and in each case measure the remaining parts :—

(i) $\angle A = 81^\circ$, $\angle B = 44^\circ$, $AB = 3{\cdot}2''$.
(ii) $BC = 6{\cdot}5$ cm., $\angle A = 63^\circ$, $\angle C = 32^\circ$.
(iii) $\angle B = 90^\circ$, $\angle C = 35^\circ$, $BC = 7{\cdot}2$ cm.
(iv) $\angle C = 112^\circ$, $\angle A = 34^\circ$, $AB = 4''$.

2. Draw a $\triangle PQR$ to the following measurements, and in each case measure the remaining parts :—

(i) $\angle Q = 76^\circ$, $\angle R = 67^\circ$, $QR = 5{\cdot}3$ cm.
(ii) $\angle Q = 99^\circ$, $\angle P = 60^\circ$, $PR = 2{\cdot}8''$.

3. Can you draw a $\triangle XYZ$ with $XY = 5''$, $\angle X = 92^\circ$ and $\angle Y = 101^\circ$? Give a reason.

4. Draw two $\triangle$'s ABC, DEF such that $BC = EF = 3{\cdot}5''$, $\angle B = \angle E = 62^\circ$, $\angle C = \angle F = 70^\circ$. Measure the remaining angles and sides of each triangle. Are the two $\triangle$'s congruent ?

5. Draw two $\triangle$'s LMN, PQR such that $\angle M = \angle Q = 115^\circ$, $\angle N = \angle R = 25^\circ$ and $LN = PR = 7$ cm. Measure the remaining parts. Are the two $\triangle$'s congruent ?

§ 6. From the above exercises we find by measurement that when two angles and a side of a triangle are given all triangles drawn from these data have the remaining corresponding sides and angle equal. Hence we are justified in assuming the truth of the following statement :—

If two triangles have two angles of the one equal to two angles of the other, each to each, and also a side of the one equal to the corresponding side of the other, the triangles are congruent.

The theoretical proof of this statement is contained in Theorem 15.

Note that if the two equal sides are not **corresponding** sides, it does not follow that the two triangles are congruent. Thus, in Fig. 85, $\angle B = \angle E$, $\angle C = \angle F$, and $AB = EF$. But AB corresponds to DE and not to EF, and it is easily seen that the triangles are not congruent.

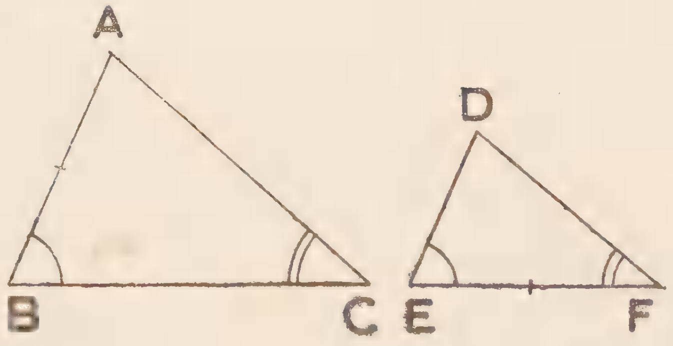


FIG. 85.

§ 7. Case III. To construct a triangle, given the three sides.

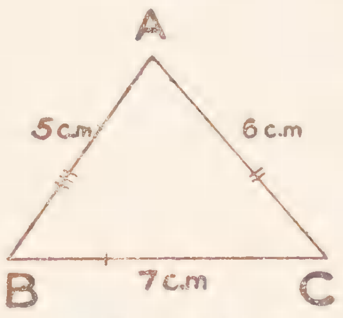


FIG. 86.

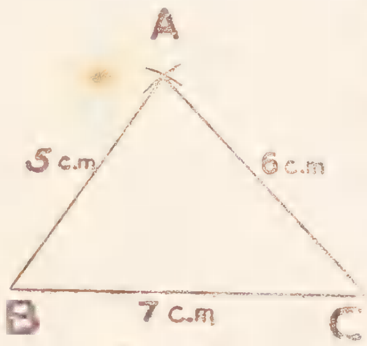


FIG. 87.

Suppose, for example, it is required to draw a triangle ABC, given that BC = 7 cm., CA = 6 cm. and AB = 5 cm.

Draw first one of the sides, BC say, making it of the required length. In this case nothing is known about the angles, so that the protractor cannot be used. The problem is to find the position of the point A. It is to be 5 cm. from B and must therefore lie **somewhere** on the circumference of a circle whose centre is B and whose radius is 5 cm. Again, the point A is to be 6 cm. from C; it must therefore lie **somewhere** on the circumference of a circle whose centre is at C and whose radius is 6 cm. Hence the required point A is where these two circles cut.

It is not necessary to draw the whole of the circumference of each circle but only the arcs near the point of intersection.

As the two circles cut at another point on the other side of BC we can draw a second triangle on BC containing the three given sides. Again, if we take the circle of radius 5 cm. with its centre at C and the circle of radius 6 cm. with its centre at B, we get two more triangles one on each side of the line BC.

EXERCISE 17

1. Draw a △ ABC to the following measurements: AB = 3·2″, BC = 3·5″, CA = 1·9″.

Having drawn the side BC first, how many such triangles can you draw? Measure the angles of each of these triangles. Are the corresponding angles equal?

2. Draw a Δ ABC to the following measurements: AB = 2″, BC = 3″, CA = 3″.

Having drawn AB first, how many such triangles can you draw in this case? Measure the angles of each triangle. Are the corresponding angles equal?

3. Can you draw a triangle with sides of the following lengths :—

(i) 4 cm., 3·5 cm., 8 cm.
(ii) 2·1″, 1·9″, 4″ ?

Give a reason for your answer.

4. Draw two Δ's PQR, XYZ to the following measurements, and in each case measure the angles :—

(i) PQ = XY = 6·2 cm., QR = YZ = 4·8 cm., RP = ZX = 5 cm.
(ii) RP = ZY = 2·4″, PQ = XY = 2″, QR = XZ = 3·6″.
(iii) QR = YZ = 8 cm., RP = ZX = 10 cm., PQ = XY = 6 cm.

§ 8. From the foregoing exercises we arrive at certain conclusions. First, if it is required to draw a triangle whose three sides are of given lengths, then, in order that this may be possible, we find that **any two sides of the triangle must be together greater than the third side.**

Next, we find by measurement that, when three sides of a triangle are given, all triangles drawn from these data have their corresponding angles equal, and the triangles, though they may differ in position, have the same size and shape. In other words, the triangles are equal in all respects or congruent. Hence we are justified in assuming the truth of the following statement :—

If two triangles have the three sides of the one equal to the three sides of the other, each to each, then the triangles are congruent.

The theoretical proof of this statement is contained in Theorem 18.

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§ 9. Case IV. To construct a triangle, given the three angles.

Suppose, for example, it is required to draw a triangle ABC, given that ∠A = 72°, ∠B = 48° and ∠C = 60°. Such a triangle is possible, for the sum of its angles is 180°.

Nothing is said in this case about the lengths of the sides. Take then a straight line of any length to represent one of the sides, BC say.

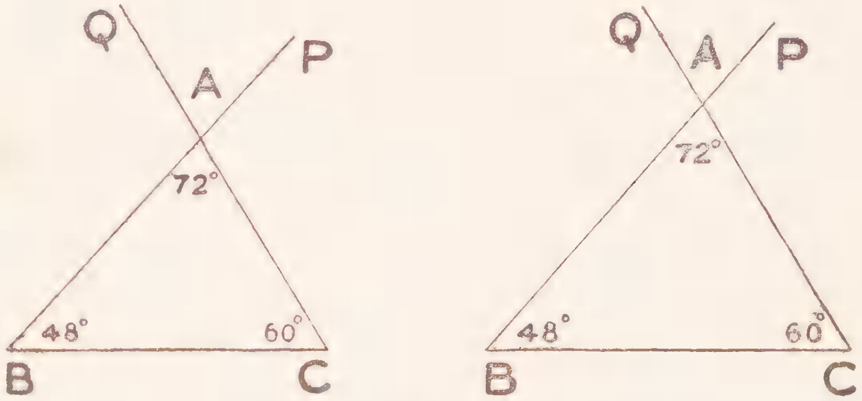


FIG. 88.

Draw at B the ∠CBP = 48° and at C the ∠BCQ = 60°. Let BP and CQ meet at A. Then ABC is a triangle containing the given angles, for we know that since the sum of the three angles is 180° the angle at A must be 72°. It is clear, as Fig. 88 shows, that any number of triangles of different sizes can be drawn to satisfy the given requirements.

Hence we are *not* justified in saying that two triangles are congruent if the three angles of the one are respectively equal to the three angles of the other.

It was stated on p. 57 that, in general, a triangle is completely determined in size and shape if three of its six parts are given. The case we have just considered appears at first sight to be an exception to the general rule. But a little reflection will show that it is not really an exception. For, if we know the value of *two* of the given angles we do not need to be told what the third angle is. We can calculate its value from the fact that the sum of the three angles is 180°. Hence we are given really only two parts of the triangle and not three.

§ 10. **Case V. To construct a triangle, given two sides and an angle which is not the included angle.**

Suppose, for example, it is required to draw a triangle ABC, given that AB = 3″, AC = 2″ and ∠ B = 34°.

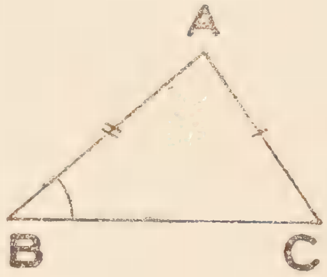


Fig. 89.

Draw a straight line BP of any length. At B draw a straight line BQ of any length making ∠ PBQ = 34°. Cut off from BQ a length BA = 3″. The problem now

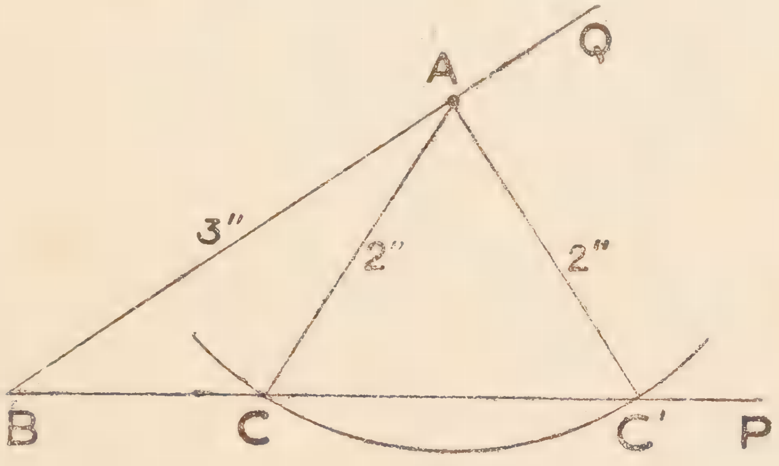


Fig. 90.

is to find the position of C. We know that C must lie somewhere on the straight line BP and that also it must be at a distance of 2″ from A. Hence with centre A and radius equal to 2″ draw an arc of a circle cutting BP. We find in this particular case that the arc cuts BP in two points. Call these points C and C′. Hence in this case we have **two** triangles ABC and ABC′, of different sizes and shapes, both of which satisfy the given conditions, namely, two sides of lengths 3″ and 2″, and an angle (not the included) of 34°.

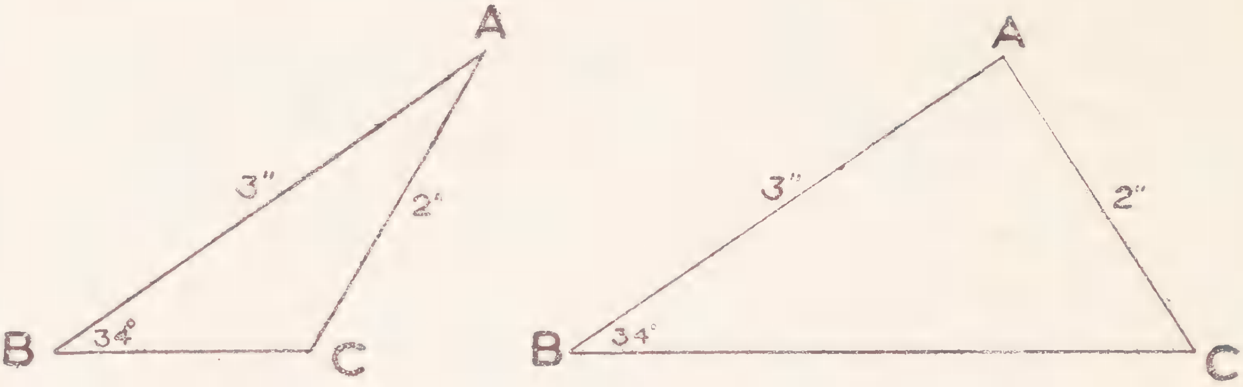


FIG. 91.

We are therefore **not** justified in saying that two triangles are congruent if two sides of the one are equal to two sides of the other and an angle of the one equal to the corresponding angle of the other, this angle being not the included angle. For, as Fig. 91 shows, it is possible to have two triangles of different sizes and shapes each of which satisfies the given conditions.

It may happen that the given side AC is of such a length that when the arc is drawn it just **touches** the line BP (Fig. 92) instead of cutting it at two points as in Fig. 90. In this case only one triangle, namely ABC, can be drawn to satisfy the given conditions, and it will be seen that AC is equal to the perpendicular drawn from A to the straight line BP.

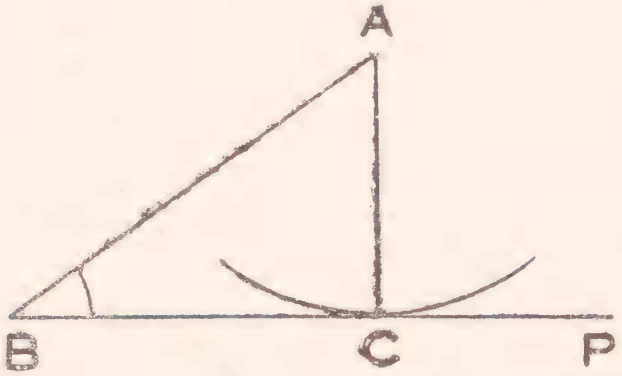


FIG. 92.

Again, it may happen that the given side AC is less than the perpendicular drawn from A to BP (Fig. 93). In this case the arc does not cut BP at any point, and therefore **no** triangle can be drawn to satisfy the given conditions.

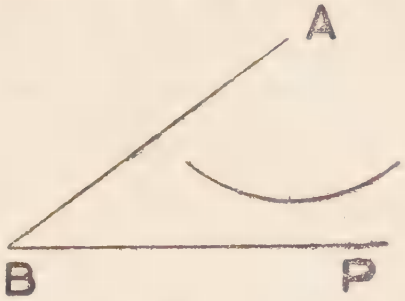


FIG. 93.

Finally, it may happen that the given side AC is **greater** than the other given side AB. In this case the arcs cut the line BP in two points C and C′ on opposite sides of B. It is clear that in this case only **one** triangle, namely ABC, satisfies the given conditions. The triangle ABC′ does not do so because, though the sides AB and AC′ are of the required length, the angle ABC′ is not equal to the given angle B but to the supplement of B.

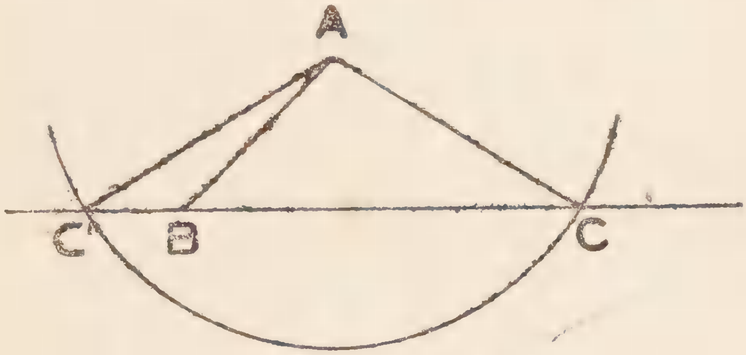


FIG. 94.

We find from the above constructions that we obtain **two** triangles only when the given angle is acute and the shorter of the two given sides is opposite the given angle.

EXERCISE 18

Construct triangles (where possible) to the following measurements, stating in each case whether only one triangle, two triangles of different sizes and shapes, or none at all can be drawn to satisfy the given conditions :—

1. CA = 3·5″, BC = 2·8″, ∠A = 50°.
2. ∠D = 105°, EF = 3·7″, FD = 2″.

3. ∠ Q = 30°, PQ = 8 cm., RP = 4 cm.
4. AB = 4″, CA = 2·3″, ∠ B = 25°.
5. XY = 8·5 cm., ∠ Y = 66°, ZX = 7·2 cm.
6. ∠ A = 54°, BC = 7·7 cm., CA = 6·5 cm.
7. ∠ P = 130°, PR = 2·8″, RQ = 2·7″.
8. ∠ A = 42°, AC = 7 cm., CB = 7 cm.
9. DE = 3·9″, EF = 2·7″, ∠ D = 36°.

§ 11. **Case VI. To construct a right-angled triangle given the hypotenuse and one side.**

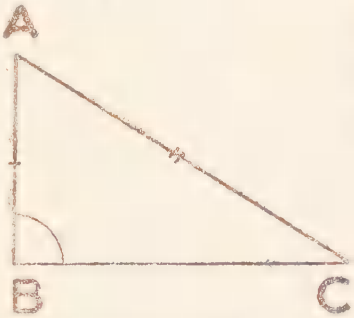


FIG. 95.

Suppose, for example, it is required to draw a right-angled triangle ABC with the right angle at B, the hypotenuse AC = 10 cm. and the side AB = 6 cm.

Draw two lines AB, BP at right angles to one another, making AB = 6 cm. and BP of any length. The point C is to be somewhere on BP and is also to be 10 cm. from A. We therefore draw, with centre A and radius equal to 10 cm., an arc of a circle cutting BP in C. Then ABC is a triangle containing the given parts.

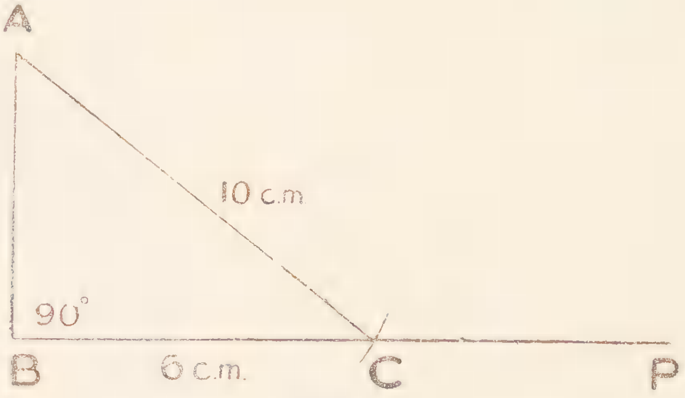


FIG. 96.

Note that Case VI is really a special case of Case V, the given angle being a right angle.

EXERCISE 19

1. Draw a △ ABC to the following measurements, and in each case measure the remaining sides and angles :—

(i) ∠ C = 90°, AB = 2·5″, BC = 1·5″.
(ii) ∠ A = 90°, AC = 5·2 cm., BC = 8·3 cm.
(iii) ∠ B = 80°, AC = 3·2″, AB = 2·2″.

2. Is it possible to draw a △ ABC with ∠ B a right angle, AB = 4″, AC = 3·5″ ?

Which is the greatest side in a right-angled triangle ?

3. Draw two △'s ABC, PQR to the following measurements :—
∠ A = ∠ P = 90°, AB = PQ = 6 cm., BC = QR = 8·5 cm.

Measure the remaining side and angles of each triangle. Are the two triangles congruent ?

§ 12. From the foregoing exercises we are led to the conclusion that when the hypotenuse and one side of a right-angled triangle are given, all triangles drawn from these data have the same size and shape. We are therefore justified in assuming the truth of the following statement :—

If two right-angled triangles have the hypotenuse and one side of the one respectively equal to the hypotenuse and one side of the other, then the triangles are congruent.

The theoretical proof of this statement is contained in Theorem 19.

SUMMARY OF RESULTS

A triangle is completely determined, that is, has **only one possible size and shape,** in the following cases :—

(1) When two sides and the included angle are given.

(2) When two angles and a side are given.

(3) When the three sides are given.

(4) When the triangle is right-angled and the hypotenuse and one side is given.

A triangle is **not** completely determined in the following cases:—

(1) When two sides and an angle which is not the included angle are given.

(2) When the three angles are given.

In the former case it has been shown that it is possible sometimes to draw two triangles of different sizes and shapes to contain the given parts, while in the latter case any number of different triangles are possible.

EXERCISE 20

Construct triangles to the following measurements, and in each case measure the remaining parts:—

1. BC = 7·7 cm., CA = 8·9 cm., AB = 5·4 cm.
2. BC = 6·9 cm., ∠B = 95°, ∠C = 43°.
3. ∠A = 90°, CA = 5 cm., BC = 7·2 cm.
4. DE = 1·75″, FD = 2·25″, ∠E = 56°.
5. AB = 6·6 cm., BC = 8·1 cm., ∠B = 32°.
6. CA = 3·5″, ∠B = 75°, ∠C = 52°.
7. DE = 3·1″, EF = 1·8″, FD = 2·4″.
8. ∠B = 36°, AB = 7·3 cm., CA = 4·5 cm.
9. ∠P = 71°, ∠Q = 49°, PR = 8 cm.
10. AB = 2·6″, BC = 3·3″, ∠A = 28°.
11. BC = 5·4 cm., ∠C = 109°, CA = 4·8 cm.
12. AB = 3·55″, BC = 2·85″, CA = 1·95″.
13. RP = 2·8″, PQ = 1·4″, ∠Q = 90°.
14. ∠X = 90°, XY = 2·6″, ZX = 1·7″.
15. ∠A = 118°, AB = 2·3″, BC = 3·7″.

ISOSCELES AND EQUILATERAL TRIANGLES

§ 13. An **isosceles** triangle has two of its sides equal.

An **equilateral** triangle has its three sides equal.

A triangle which has no two sides equal is called **scalene.**

Draw a triangle ABC to the following measurements :— BC = 2·3″, AB = AC = 3·4″. Such a triangle is isosceles with AB, AC as the two equal sides. Measure the angles of this triangle. Repeat this exercise, drawing other isosceles triangles with sides of different lengths. We find that in all such triangles the angles opposite the equal sides are equal. This is only what we should expect, for there seems to be no reason why one angle should be greater than the other.

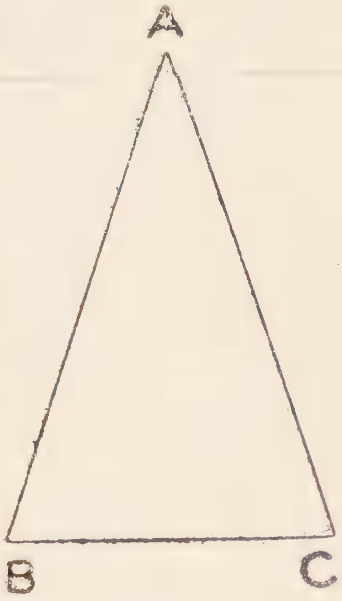


FIG. 97.

Draw now a triangle with two of its angles equal. For example, draw a triangle ABC with BC = 6 cm., ∠ B = ∠ C = 68°. Measure its sides.

Repeat this exercise with different measurements. In each case we find that the sides which are opposite the equal angles are equal. Again this is only what we should expect.

From these exercises we are led to accept the truth of the following statements :—

(1) **If two sides of a triangle are equal, the angles opposite those sides are equal.**

(2) **If two angles of a triangle are equal, the sides opposite those angles are equal.**

The theoretical proofs of these statements are contained in Theorems 16 and 17.

§ 14. Draw any straight line AB. With centre A and radius equal to AB draw an arc of a circle. With centre B and the same radius draw another arc cutting the former in **C**. Then **ABC** is an equilateral triangle.

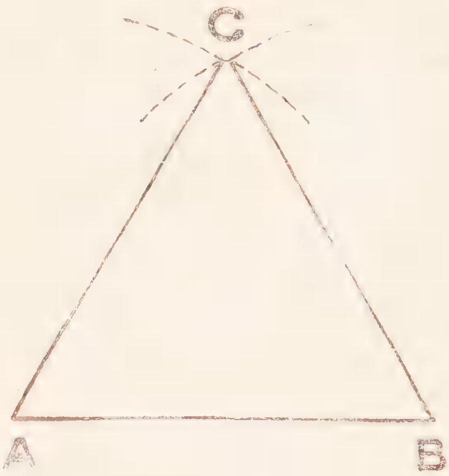


FIG. 98.

Measure the angles of the triangle. We find that all the three angles are equal. This, of course, follows from § 13.

What is the value of each angle? How could you have arrived at this fact without making any measurements?

Exercises on the construction of isosceles triangles will be found after Theorems 16 and 17 (Exercises 26 and 27).

CONSTRUCTION OF QUADRILATERALS

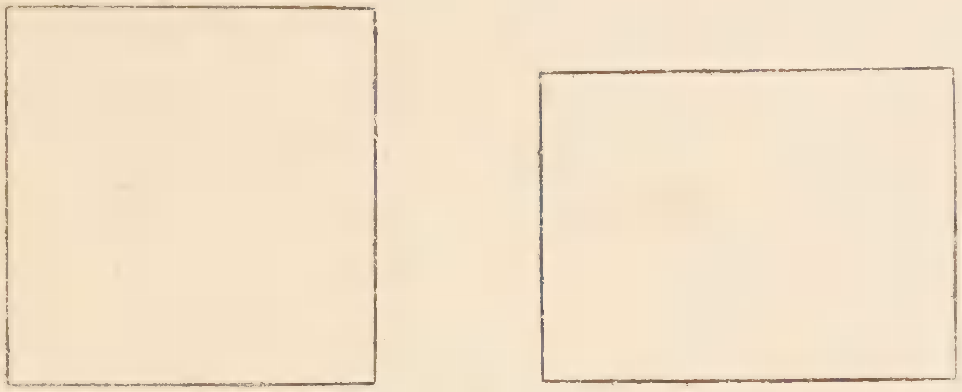
§ 15. We have seen that, in general, a triangle is completely determined when three of its parts are given. It will be found that in order to determine a quadrilateral completely, five of its parts must be given.

The quadrilaterals in the following exercises can be drawn by methods similar to those used in the construction of triangles. When a diagonal is given the problem is often reduced to constructing two triangles each of which has three parts given. The fact that the sum of the angles of a quadrilateral is 360° can be used to check the measurement of angles.

It is important to draw first a rough freehand sketch of the quadrilateral and mark on it the given parts.

When all the sides are equal and each of the angles is a right angle, the quadrilateral is called a **square**, as has already been mentioned (p. 51).

When the opposite sides are equal and each of the angles is a right angle, the quadrilateral is called a **rectangle** or **oblong.**



Square. Rectangle.

FIG. 99.

EXERCISE 21

Draw quadrilaterals to the following measurements, and in each case measure the remaining sides and angles. When compasses are used, the construction lines (that is, arcs of circles) should be shown clearly :—

1. ABCD, given AB = 3·2″, BC = 2·6″, CD = 1·8″, DA = 2·5″, ∠ A = 67°.

Measure the diagonal BD. Would it be possible to draw a quadrilateral ABCD with AB, DA and ∠ A the same as before but with BC = 1·6″ and CD = 1·2″ ?

2. ABCD, given AB = 7 cm., AD = 4·5 cm., ∠ A = 123°, ∠ B = 77°, ∠ D = 80°.

3. ABCD, given BC = 2″, CD = 3·4″, ∠ A = 105°, ∠ B = 121° and ∠ C = 84°.

4. ABCD, given AB = 8·2 cm., BC = 6 cm., CD = 6·6 cm., ∠ A = 69°, ∠ B = 90°.

Can you draw more than one kind of quadrilateral to satisfy the given conditions? (Compare Case V in the construction of triangles, p. 65.)

5. ABCD, given AB = 2·2″, CD = 3·5″, DA = 2·4″, ∠ A = 64°, ∠ B = 135°.

6. PQRS, given PQ = 3·3″, QR = 2″, RS = 1·8″, SP = 2·4″, diagonal QS = 3·2″.

7. ABCD, given AB = 5 cm., BC = 7·6 cm., diagonal AC = 11·5 cm., ∠ BAD = 67°, ∠ BCD = 42°.

8. DEFG, given DE = 6·4 cm., EF = 8·3 cm., FG = 5·5 cm., diagonal EG = 9·5 cm., ∠ FGD = 90°.

Can you draw more than one kind of quadrilateral in this case which will satisfy the given conditions ?

9. ABCD, given AB = 2·5″, BC = 3·2″, CD = 2·2″, diagonal AC = 3″, diagonal BD = 4″.

10. Draw a square of side 2·3″. Measure the diagonals and also the angles which the diagonals make with the sides.

11. Draw a square a diagonal of which is 6·8 cm.

12. Draw a rectangle with adjacent sides 7 cm. and 5·5 cm. long. Measure the diagonals.

13. Draw a rectangle given one side 3·2″ and the angle between this side and a diagonal 40°.

14. Draw a rectangle with one side 2·5″ and a diagonal 3″.

CHAPTER VII

DRAWING TO SCALE (FURTHER EXERCISES)

§ 1. When a weight is tied to the end of a string and allowed to hang freely at rest, the direction taken by the string is called the **vertical** direction and is the direction in which the weight falls if the string is let go.

Any straight line at right angles to a vertical line is said to be **horizontal** or level.

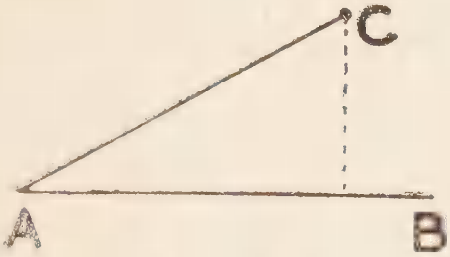


FIG. 100.

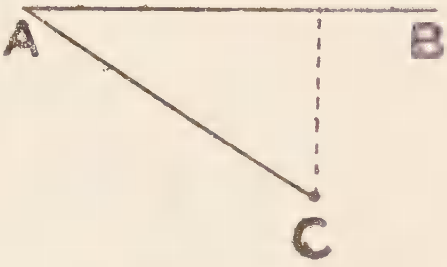


FIG. 101.

Let AB be a horizontal straight line and let C be the position of some object. If the perpendicular from C on AB is vertical, then in Fig. 100 C is said to be **vertically above** AB and in Fig. 101, C is said to be **vertically below** AB.

In Fig. 100, $\angle$ BAC is called the **angle of elevation** of the object C as seen from the point A.

In Fig. 101, $\angle$ BAC is called the **angle of depression** of the object C as seen from the point A.

In the former case the line AB is turned through an angle which elevates or raises it from the horizontal position AB to the position AC. In the latter case AB is depressed or lowered through an angle which brings it from the horizontal position to the position AC.

The angle BAC is sometimes referred to as the angle **subtended** at A by the points B and C, or by the straight line BC.

In some of the following exercises triangles have to be constructed by the methods of the previous chapter. Draw, as before, a freehand sketch of the required figure and mark on it all the given parts. Then make the accurate drawing, choosing the most suitable scale (when no scale is given). The construction lines should not be rubbed out, for they show how the figure has been made.

EXERCISE 22

1. A weight is suspended by means of two strings of lengths 5 ft. and 8 ft. from two hooks fixed in a horizontal beam. If the distance between the hooks is 6 ft., find the angle which each string makes with the horizontal.

Find also the vertical distance of the weight below the beam. [Scale 1 cm. to 1 foot.]

2. A, B and C are three villages. The distance from A to C is 18 miles, and from A to B 32 miles. If the angle subtended by B and C at A is 105°, find the distance between B and C.

3. A ladder 20 ft. long just reaches the top of a vertical wall. If the foot of the ladder is on level ground at a distance of 6 ft. from the foot of the wall, find the height of the wall and also the angle the ladder makes with the ground. [Scale 1 in. to 5 ft.]

4. A house H is observed from two points A, B on a straight horizontal road. If $AB = \frac{1}{2}$ mile, $\angle BAH = 72^\circ$, and $\angle ABH = 33^\circ$, find the distance of the house from the points A, B and also its distance (that is, perpendicular distance) from the road. [Scale 8 in. to 1 mile.]

5. A gun is 20 miles away from a straight road and its range (that is, the greatest distance to which it can fire a shell) is 25 miles. What length of road can be shelled by the gun?

6. The sides of a triangular field are 120 yds., 95 yds. and 180 yds. long. Find the angles of the field and also the perpendicular distance of each corner from the opposite side. [Scale 1 in. to 50 yds.]

7. A plank 15 ft. long is placed with one end resting against a vertical wall at a point which is 9 ft. above the horizontal ground. Find the distance of the foot of the plank from the foot of the wall.

8. A, B are two posts on the bank of a straight river near its edge. C is a tree on the edge of the opposite bank. If ∠ ABC = 47°, ∠ BAC = 35° and AB = 360 ft., find the breadth of the river.

9. A swing is 15 ft. long, and when at rest the seat of the swing is 3 ft. above the ground. Find the height of the seat above the ground when the swing makes an angle of (i) 30°, (ii) 60° with the vertical.

10. Jhansi, Allahabad and Benares, taken in order are in a straight line. The distance from Jhansi to Allahabad is 200 miles, and from Allahabad to Benares is 72 miles. Cawnpore is 128 miles from Jhansi and 184 miles from Benares. Draw a plan to a scale of 1 inch to 80 miles, and find from it the distance of Cawnpore from Allahabad.

11. A four-sided field ABCD is of the following dimensions: AB = 320 yds., BC = 190 yds., CD = 150 yds., DA = 330 yds., and ∠ A = 56°. Draw a plan of the field and find from it the lengths of the diagonals AC and BD.

12. The angle of elevation of the top of a monument as observed from a point 200 ft. away in the horizontal plane through the base of the monument is 60°. Find the height of the monument.

13. From the top of a light-house 150 ft. above the level of the sea the angle of depression of a boat is found to be 24°. How far is the boat from the light-house?

14. At a point on the ground 480 ft. away from the foot of a wireless mast the angle of elevation of the top of the mast is 42°. Find the height of the mast.

15. At a certain place I find the angle of elevation of the top of a hill to be 18°. On walking a mile towards the hill along a straight horizontal road I find the angle of elevation to be 27°. Find, to the nearest 100 ft., the height of the hill above the level of the road.

16. A hill stands 2,300 ft. high above the level of a plain. A person on the top of the hill observes two villages in the plain due south of him. If their angles of depression are 28° and 30°, find their distance apart.

17. A toy-balloon is attached by means of a string 35 ft. long to the top of a post 12 ft. high. If the angle of elevation of the balloon is 55°. find its height above the ground.

18. An observer from the top of a mast 90 ft. above sea-level finds the angles of depression of two buoys, one due north and the other due south, to be 33° and 58°. Find the distance between the buoys.

19. A tower is 160 ft. high; an observer in a balloon finds the angles of depression of the top and bottom of the tower to be 31° and 47°. Find the height of the balloon above the ground.

20. A person standing at a point A finds that the angle of elevation of the top of the Shwe Dagon Pagoda is 20°. He then walks towards the Pagoda a distance of 920 ft. along a slope which is inclined at an angle of 12° to the horizontal, arriving at a point, B on the upper terrace. At B he finds the angle of elevation to be 28°. Find the height of the top of the pagoda above (i) the upper terrace, (ii) the horizontal plane through A [Scale 1 in. to 400 ft.]

POINTS OF THE COMPASS—(*continued*)

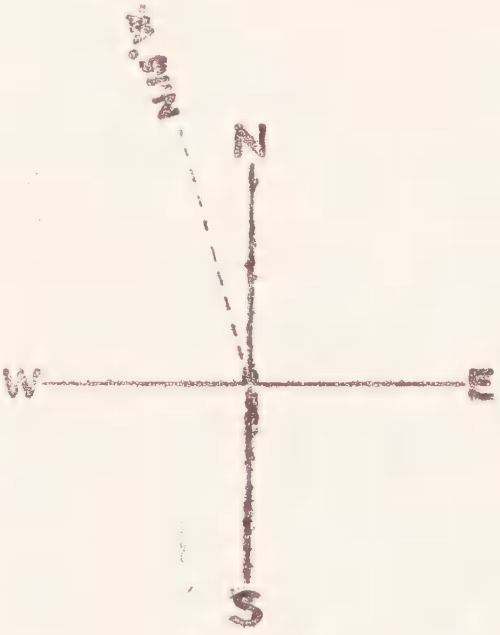


FIG. 102.

§ 2. If you face to the north and then turn through an angle of 15° towards the west, the direction in which you are now facing is said to be '15° West of North' or 'N. 15° W.'

If you are looking at some object in that direction the object is said to **bear** N. 15° W, or has a **bearing** of N. 15° W, from the place at which you are standing.

Example. The bearing of B from A is S. 66° E., and the bearing of C from A is N. 28° E. If AB is 7 miles and AC 4 miles, find the distance and bearing of C from B.

First draw through A the N. and S. and E. and W. lines. Let us take a scale of 1 inch to 2 miles. Draw AB and AC in the required directions, making AB = 3·5″ and AC = 2″. To obtain the bearing of C from B draw through B a N. and S line. It will be found that

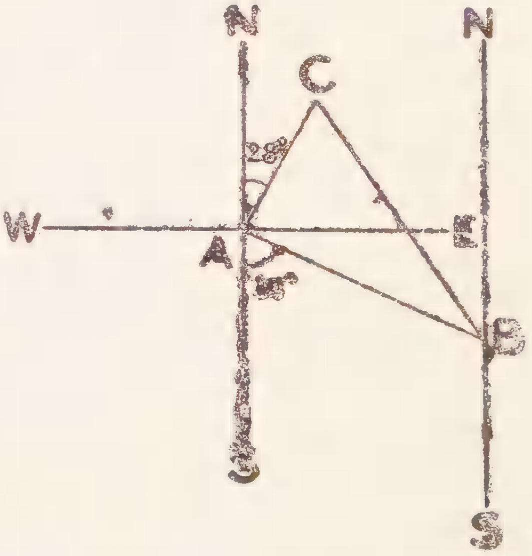


FIG. 103.

BC makes with this line an angle of 36° and that the length of BC is approximately 4″.

Hence the distance of C from B is approximately 8 miles and its bearing from B is N. 36° W.

EXERCISE 23

1. Dacca is 152 miles N. 58° E. of Calcutta, and Asansol is 216 miles due W. of Dacca. Find the distance and bearing of Asansol from Calcutta. [Scale 1 in. to 80 miles.]

2. Pegu is 45 miles N. 25° E. of Rangoon, and Moulmein is 100 miles S. 78° E. of Rangoon. Find the distance and bearing of Pegu from Moulmein.

3. A person standing at a place A observes two temples B and C. He finds that B bears N. 32° W. and is distant from him 2,400 yards, and that C bears N. 44° E. and is distant from him 3,800 yards. Find the distance and bearing of C from B.

4. Cawnpore is 240 miles S. 52° E. of Delhi. Gwalior is 180 miles from Delhi and 140 miles from Cawnpore and southward of both towns. Find the bearing of Gwalior from Delhi.

5. Lucknow is 260 miles N. 15° E. of Jubbulpore, and Allahabad is 220 miles N. 27° E. of Jubbulpore. Find the distance and bearing of Lucknow from Allahabad.

6. A ship sailing at the rate of 12 miles an hour sails for 1½ hr. in a direction S. 35° W. She then alters her course to S. 70° W. Find her distance and bearing from the starting-point at the end of another 2 hours.

7. Mandalay is 95 miles due N. of Thazi. Taunggyi is due E. of Thazi and at a distance of 100 miles from Mandalay. Find the bearing of Taunggyi from Mandalay.

8. Ahmedabad is 290 miles N. 5° W. of Bombay. Karachi is 340 miles N. 72° W. of Ahmedabad. Find the distance and bearing of Karachi from Bombay.

9. Ferozepore, Ludhiana, Amritsar and Lahore, in that order. form the corners of a quadrilateral. Ludhiana is 72 miles due E. of Ferozepore, and Amritsar is 76 miles N. 49° W. of Ludhiana. Lahore is 48 miles from Ferozepore and 32 miles from Amritsar and westward of both towns. Draw a plan to a scale of 1 in. to 20 miles and find from it the distance and bearing of Amritsar from Ferozepore.

CHAPTER VIII

CONGRUENCE OF TRIANGLES

§ 1. It has already been stated that by two **congruent** triangles is meant two triangles equal in all respects, namely, the three sides of the one equal to the three sides of the other, and the three angles of the one equal to the corresponding three angles of the other.

If two triangles can be placed, one on top of the other, in such a way that they **coincide**, that is, fit each other exactly, it is obvious that the two triangles are congruent. The three sides of the one will necessarily be equal to the three sides of the other and the three angles of the one equal to the corresponding three angles of the other; also the two triangles will have equal areas.

In Chapter VI a number of triangles were constructed to given measurements. It was found that in certain cases, when three of the six parts of a triangle are given, all triangles drawn to these measurements are congruent. We shall now give theoretical proofs of the results which we arrived at experimentally. In some theorems the method of proof will be by **superposition**, that is, placing one triangle on top of the other and showing that they can be made to coincide.

Note.—In order to make these proofs clearer the teacher may cut out triangles of a fair size from a piece of cardboard. Thus in proving Theorem 14 cut out a $\triangle$ DEF. Then cut out (1) a $\triangle$ ABC having AB = DE, AC = DF, but $\angle$ BAC not equal to $\angle$ EDF, (2) a $\triangle$ ABC having AB = DE, $\angle$ BAC = $\angle$ EDF, but AC not equal to DF, and (3) a $\triangle$ ABC having AB = DE, AC = DF, and $\angle$ BAC = $\angle$ EDF. By placing each of these three $\triangle$'s ABC in turn on the $\triangle$ DEF with AB placed on DE, the points of the argument can be clearly followed. Similarly for Theorem 15.

*THEOREM 14

§ 2. **If two triangles have two sides of the one equal to two sides of the other, and the angles included by those sides equal, the triangles are congruent.**

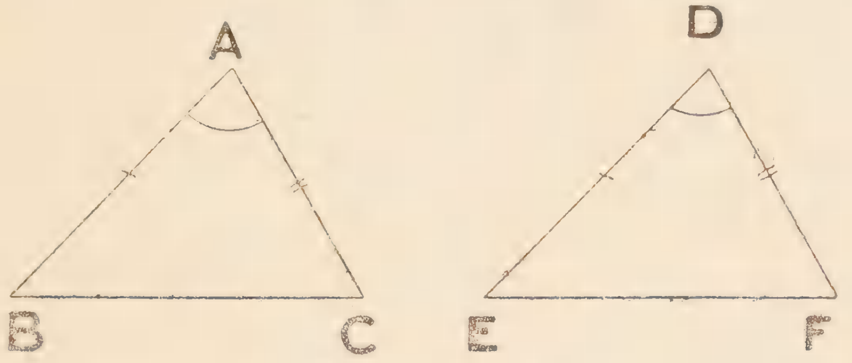


FIG. 104.

Given two triangles ABC, DEF in which AB = DE, AC = DF and ∠ A = ∠ D.

To prove that the triangles ABC, DEF are congruent.

Proof. Place the △ ABC on the △ DEF in such a way that the point A falls on the point D and the side AB along the side DE.

Then, because AB = DE,

the point B falls on the point E.

And, because AB falls on DE and ∠ A = ∠ D, AC falls on DF and therefore C lies somewhere on the line DF.

But, because AC = DF,

the point C falls on the point F.

Thus the points A, B, C fall on the points D, E, F and the two triangles fit each other exactly.

Hence the two triangles are congruent, and the remaining corresponding parts are equal, namely, BC = EF, ∠ B = ∠ E, and ∠ C = ∠ F. Also the areas of the triangles are equal.

Note.—Theorem 14 proves the result arrived at in Case I of Construction of Triangles (p. 57).

EXERCISE 24

The following riders are to be proved by means of Theorem 14 :—

1. ABC is a triangle having ∠ ABC = ∠ ACB. BD is drawn perpendicular to AB, and CE perpendicular to AC. If BD = CE, prove that BE = CD.

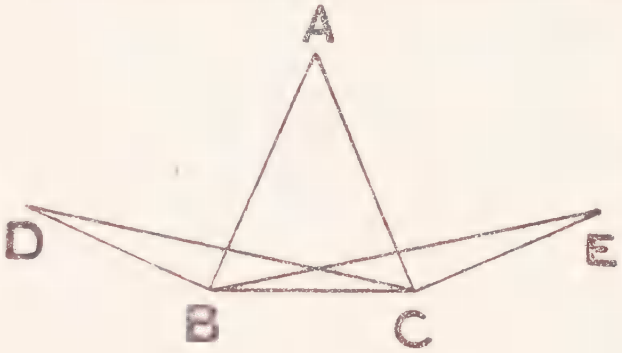


FIG. 105.

[Our object here is to find two congruent Δ's which contain BE and CD as corresponding sides. Consider the Δ's DBC, ECB. DB is given equal to EC. Also BC is common, that is, belongs to both Δ's. It remains to show that the included ∠ DBC = the included ∠ ECB. We are given that ∠ ABC = ∠ ACB. Also ∠ ABD = ∠ ECA each being a right angle, for we are given that BD is perpendicular to AB, and CE to AC. Hence the whole ∠ DBC = the whole ∠ ECB. We can therefore apply the result of Theorem 14. The proof may be written out as follows :—

∠ ABC = ∠ ACB
∠ ABD = ∠ ACE, each being a right angle.
∴ the whole ∠ DBC = the whole ∠ ECB.
Hence, in the Δ's DBC, ECB
DB = EC,
BC is common,
the included ∠ DBC = the included ∠ ECB.
∴ the Δ's DBC, ECB are congruent
∴ CD = BE].

2. AOB, BOC are two equal angles and OA = OC. Prove that BA = BC.

3. ABC is an isosceles Δ with AB, AC as the two equal sides. P and Q are points in AB and AC respectively such that AP = AQ. Prove that BQ = CP.

4. AB is a straight line and M its middle point. A straight line MN is drawn perpendicular to AB. Prove that ANB is an isosceles Δ.

5. P, Q, R, S are the middle points of the sides AB, BC, CD, DA of a square ABCD. Prove that PQ = QR = RS = SP.

6. Two straight lines AB and CD are both bisected at a point O. Prove that (i) AC = BD and AD = BC, and (ii) AC is parallel to BD, and AD parallel to BC.

*THEOREM 15

§ 3. **If two triangles have two angles of the one equal to two angles of the other, each to each, and any one side of the one equal to the corresponding side of the other, the triangles are congruent.**

(*a*) **Let the two given angles be adjacent to the given side.**

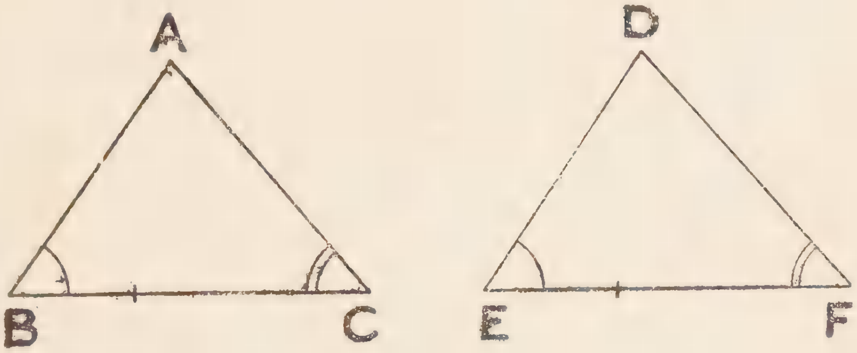


FIG. 106.

Given the two triangles ABC, DEF in which ∠ B = ∠ E, ∠ C = ∠ F and BC = EF.

To prove that the triangles ABC, DEF are congruent.

Proof. Place the △ ABC on the △ DEF so that the point B falls on the point E and BC along the side EF.

Then, because BC = EF, the point C falls on the point F.

And, because ∠ B = ∠ E, BA falls along ED and therefore A lies somewhere on ED or ED produced.

Again, because ∠ C = ∠ F, CA falls along FD and therefore A lies somewhere on FD or FD produced.

∴ A must fall on the point D where ED and FD meet.

Thus the points A, B, C fall on the points D, E, F and the two triangles coincide.

Hence the triangles are congruent, and the remaining parts are equal, namely, ∠ A = ∠ D, AB = DE, AC = DF. Also the triangles are equal in area.

(*b*) **Let only one of the given angles be adjacent to the given side.**

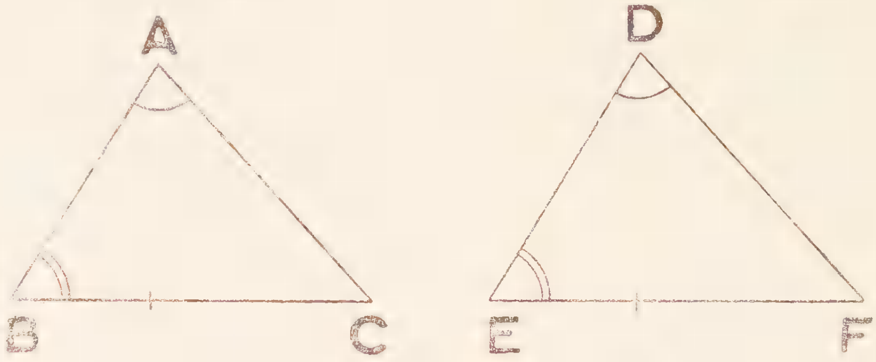


FIG. 107.

Given the two triangles ABC, DEF in which $\angle A = \angle D$, $\angle B = \angle E$ and BC = EF.

To prove that the triangles ABC, DEF are congruent.

Proof. The sum of the angles of a triangle is two right angles.

$\therefore \angle A + \angle B + \angle C = \angle D + \angle E + \angle F$.

But it is given that $\angle A = \angle D$ and $\angle B = \angle E$.

$\therefore \angle C = \angle F$.

Hence the triangles are congruent by what has already been proved in Case (*a*).

Note.—Theorem 15 corresponds to Case II of Construction of Triangles (p. 59). When one of the two given angles was not adjacent to the given side, it was found impossible to construct the triangle without first calculating the value of the third angle. This was done by making use of the fact that the sum of the angles of a triangle is 180°.

We meet with a similar difficulty in the theoretical proof. For, suppose, we attempt to prove the congruence of the two triangles ABC, DEF in Fig. 107 without first proving that $\angle C = \angle F$. As before, because BC = EF, we can place the point B on the point E, and the point C on the point F. And, because $\angle B = \angle E$, BA falls along ED. But, we cannot say that the point A falls on the point D, for we are not given AB = DE. Nor can we say that CA will fall along FD, for we are not given $\angle C = \angle F$. Hence a proof by superposition is not possible unless we show first that the second adjacent angle C of the triangle ABC is equal to the second adjacent angle F of the triangle DEF.

EXERCISE 25

1. In a △ ABC the bisector of the ∠ A is perpendicular to BC. Prove that the triangle is isosceles.

2. In a △ ABC, AB = AC. Prove that the perpendiculars drawn from B and C to the opposite sides are equal.

3. P is any point on the bisector of an ∠ BAC. Prove that the perpendiculars drawn from P to AB, AC are equal.

4. PQ is a straight line drawn through C, the middle point of another straight line AB. D and BE are drawn perpendicular to PQ. Prove that (i) AD = BE, (ii) AE is equal and parallel to BD.

5. In a quadrilateral ABCD, AB is equal and parallel to CD; prove that AC and BD bisect each other.

THEOREM 16

§ 4. **If two sides of a triangle are equal, the angles opposite these sides are equal.**

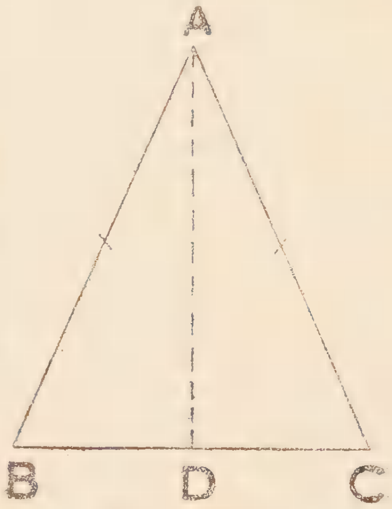


FIG. 108.

Given that ABC is a triangle in which AB = AC.

To prove that ∠ B = ∠ C.

Suppose AD is the straight line which bisects ∠ BAC, and let it meet BC in D.

Proof. In the △'s ABD, ACD, AB = AC.

The side AD is common to both triangles.

And the included ∠ BAD = the included ∠ CAD.

∴ the triangles are congruent.

∴ ∠ B = ∠ C.

The point in which the two equal sides of an isosceles triangle meet is called the **vertex** of the triangle, the angle at that point is called the **vertical angle**, and the side opposite the vertex is called the **base**. Thus, in Fig. 108, A is the vertex, BAC the vertical angle, and BC the base. The angles at B and C are called the base angles. The straight line AD drawn from the vertex perpendicular to the base is called the **altitude,** or height, of the triangle.

Theorem 16 can be stated in the following form :—

The angles at the base of an isosceles triangle are equal.

Cor. The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

For, it has been proved above that the Δ's ABD, ACD are congruent. ∴ BD = CD and ∠ ADB = ∠ ADC. But the sum of these two angles is 2 right angles. ∴ each is a right angle. ∴ AD is perpendicular to BC and also bisects it.

Note.—In the proof of Theorem 16 given above we have made use of a theorem already proved, namely, Theorem 14. The theorem can, however, be proved **directly,** that is, without making use of a previously known theorem.

Let AD, as before, be the bisector of ∠ BAC. Suppose the triangle is now folded about AD. Then, because ∠ BAD = ∠ CAD, the side AB will fall along the side AC and therefore B will lie somewhere on AC. But, because AB = AC, the point B will fall on the point C. Hence, when the folding takes place, we have the points A and D common to the two triangles ABD, ACD and the point B falls on the point C. The two triangles then coincide and are therefore congruent.

When a figure can be folded about a straight line so that the two parts of the figure coincide, the figure is said to be **symmetrical** about that line. Thus in Fig. 108, the Δ ABC is symmetrical about the line AD.

EXERCISE 26

1. Prove that an equilateral triangle is also equiangular. Hence calculate the number of degrees in an angle of an equilateral triangle.
2. Draw with ruler and compasses only an angle of 60°.
3. Draw an isosceles triangle with base 3″ and altitude 3·5″.
4. Prove that the angle between a diagonal and a side of a square is 45°.
5. If the base of an isosceles triangle be produced both ways, prove that the exterior angles thus formed are equal.

6. ABC is any triangle with an acute angle at A. Equilateral triangles ABD, ACE are drawn on AB, AC. Prove that ∠ BAE = ∠ CAD.

7. RPQ, SPQ are two isosceles triangles drawn on opposite sides of the same base PQ. Prove that ∠ RPS = ∠ RQS.

8. Find the angles of an isosceles triangle in which each of the base angles is (i) double, (ii) four times, the angle at the vertex.

9. Prove that the straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

10. One of the equal sides of an isosceles triangle is produced beyond the vertex. Prove that the bisector of the exterior angle thus formed is parallel to the base.

THEOREM 17

§ 5. **If two angles of a triangle are equal, the sides opposite those angles are equal.**

Given that ABC is a triangle in which ∠ B = ∠ C.

To prove that AB = AC.

Suppose AD is the straight line which bisects ∠ BAC and let it meet BC in D.

Proof. In the triangles ABD, ACD, ∠ ABD = ∠ ACD.

And, because AD bisects ∠ BAC, ∠ BAD = ∠ CAD.

Also, the side AD is common to both triangles.

∴ the triangles ABD, ACD are congruent.

∴ AB = AC.

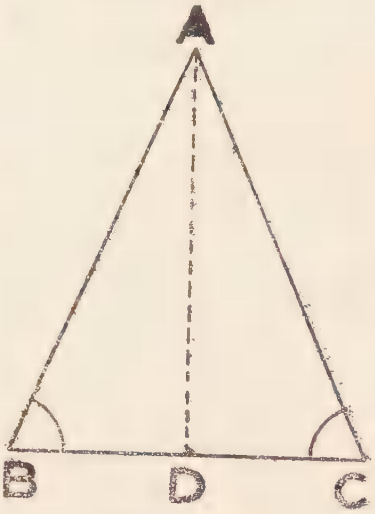


FIG. 109.

EXERCISE 27

1. Which previous theorem has been used to prove Theorem 17?

2. Show that it is not possible to give a direct proof of Theorem 17 by folding the △ ABC about the line AD. Explain where the proof breaks down.

3. Prove that if a triangle is equiangular it is also equilateral.

4. Construct an isosceles triangle whose base is 2·5″ and whose vertical angle is (i) 30°, (ii) 60°. [First calculate the base angles.]

5. Construct an isosceles triangle whose base is 6·5 cm. and whose vertical angle is (i) 44°, (ii) 118°.

6. Prove that the bisectors of the base angles of an isosceles triangle form with the base a triangle which is isosceles.

7. Prove that a straight line drawn parallel to the base of an isosceles triangle forms with the other two sides a triangle which is isosceles.

8. The base **BC** of an isosceles △ **ABC** is produced both ways to **D** and **E**. The bisectors of the exterior angles **ABD, ACE** meet at **O**. Prove that **OBC** is an isosceles triangle.

NOTE ON CONVERSE THEOREMS

§ 6. In Theorem 16 it is given that two sides of a triangle are equal and it is proved that the two angles opposite those sides are equal. In Theorem 17 on the other hand, it is given that two angles of the triangle are equal, and it is proved that the two sides opposite those angles are equal. When two theorems are so related that what is given in the one is proved in the other, they are said to be **Converse theorems.**

The above two converse theorems both happen to be true. Theorems 5 to 10 on parallels also provide examples of pairs of converse theorems both of which are true. But it does not always follow that when a statement is true the converse statement also is true. For example, ‘If rain has fallen, the grass is wet’ may be a true statement. But the converse statement, namely, ‘If the grass is wet, rain has fallen’ is not necessarily true. The grass may be wet for some other reason. For instance, it may have been watered by the ‘mali’.

Give the converse of the following statements and say in each case whether it is true or not :—

(i) If a triangle is equilateral, it is also equiangular.

(ii) If the boy weeps, he is unhappy.

(iii) If a straight line cuts two other straight lines so that the alternate angles are equal, the two straight lines are parallel.

(iv) If one angle of a triangle is obtuse the other two angles are acute.

(v) If two triangles have three sides of the one equal to three sides of the other, each to each, the three angles of the one are equal to the corresponding three angles of the other.

(vi) When I sleep my eyes are closed.

*THEOREM 18

§ 7. If two triangles have three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

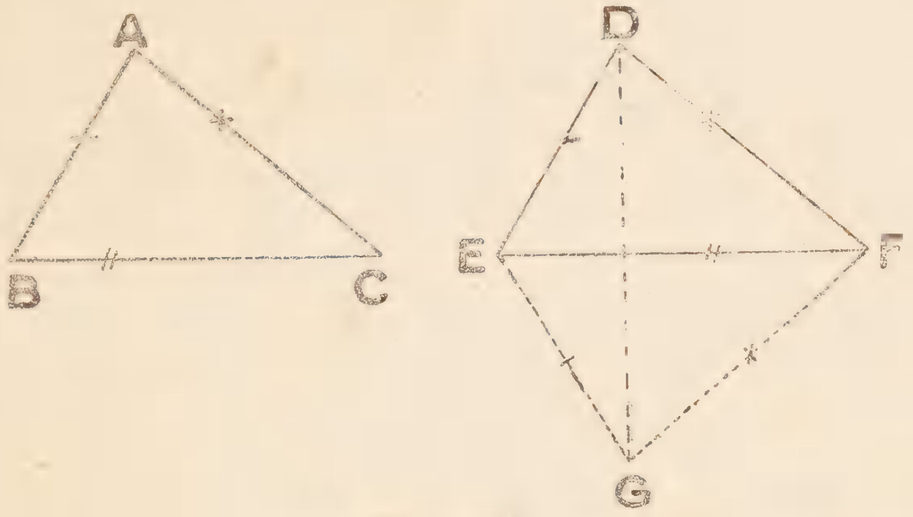


FIG. 110.

Given two triangles ABC, DEF in which AB = DE, BC = EF, and CA = FD.

To prove that the triangles are congruent.

Proof. Suppose BC, EF to be the greatest sides. Then, because BC = EF, the △ ABC can be placed in such a way that BC coincides with EF and A lies on the other side of EF opposite to D.

Let GEF be the new position of the △ ABC.

Then, GE = AB = DE.

∴ ∠ EDG = ∠ EGD, being the angles opposite two equal sides of a triangle.

Again, GF = AC = DF.

∴ ∠ FDG = ∠ FGD,

∴ the whole ∠ EDF = the whole ∠ EGF

= ∠ BAC.

Hence, in the triangles ABC, DEF

AB = DE.

AC = DF.

and the included ∠ BAC = the included ∠ EDF.

$\therefore$ the triangles are congruent, and the corresponding angles are equal, namely, $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$; also the areas of the triangles are equal.

Note.—1. The greatest side BC is taken for superposition in order that the line DG may fall within the angles EDF, EGF.

2 Theorem 18 proves the result arrived at in Case III, of Construction of Triangles (p. 62).

EXERCISE 28

1. What two theorems are used in the proof of Theorem 18 given above?

2. Explain why it is not possible to prove Theorem 18 by direct superposition as in Theorems 14 and 15.

3. In a quadrilateral ABCD, AB = AD and CB = CD; prove that AC bisects the angles at A and C.

4. Prove that the straight line joining the vertex of an isosceles triangle to the middle point of the base bisects the vertical angle.

5. If the opposite sides of a quadrilateral are equal, prove that they are also parallel. [Prove that the two triangles formed by drawing a diagonal are congruent, and use one of the theorems on parallels.]

THEOREM 19

§ 8. **If two right-angled triangles have the hypotenuse and one side of the one equal respectively to the hypotenuse and one side of the other, the triangles are congruent.**

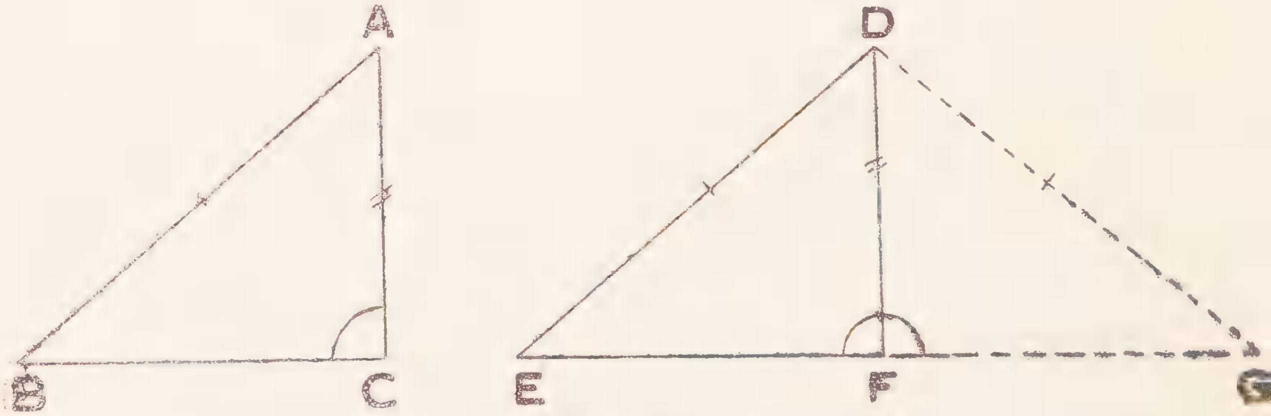


FIG. 111.

Given two triangles ABC, DEF in which the angles at C and F are right angles, hypotenuse AB = hypotenuse DE, and the side AC = the side DF.

To prove that the triangles ABC, DEF are congruent.

Proof. Because AC = DF, the △ ABC can be placed in such a way that AC coincides with DF and B lies on the side of DF opposite to E.

Let DGF be the new position of the △ ABC.
∠ DF~~A~~G, that is, ∠ ACB = a right angle.
And DFE = a right angle.
∴ EF and FG are in the same straight line.
In the △ DEG, DE = DG (that is, AB)
∴ ∠ E = ∠ G.
Hence, in the triangles DEF, DGF,
∠ DFE = ∠ DFG, being right angles,
∠ E = ∠ G
and the side DF is common

∴ the triangles DEF, DGF are congruent, that is, the triangles DEF, ABC, are congruent and therefore the remaining parts are equal, namely, BC = EF, ∠ A = ∠ D, and ∠ B = ∠ E ; also the areas of the triangles are equal.

Note.—1. In the proof of Theorem 19 use is made of three previous theorems. It is necessary first to use Theorem 3 to prove that EF and FG are in the same straight line, otherwise we are not justified in taking the ∠ DEF to be the same as the ∠ DEG, as Fig. 112 makes clear. In the next place, Theorem 16 is used to prove that the angles at E and G are equal. Finally, having shown that the two triangles have two angles and a side of the other we know by Theorem 15 that the triangles are congruent.

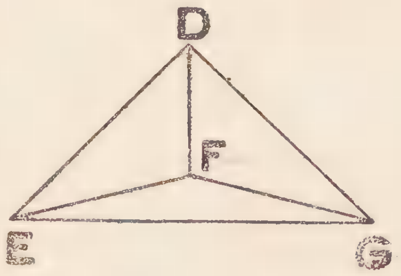


FIG. 112.

2. Theorem 19 corresponds to Case VI of Construction of Triangles (p. 67).

EXERCISE 29

1. Prove that the perpendicular drawn from the vertex to the base of an isosceles triangle bisects both the base and the vertical angle.

2. In a quadrilateral ABCD the angles at B and D are right angles and AB = AD. Prove that AC bisects the angles at A and C.

3. Prove that the middle point of the base of an isosceles triangle is equidistant from the equal sides (that is, prove that the perpendiculars from the middle point to the equal sides are equal).

4. A point within a given angle is equidistant from the arms of the angle; prove that the line joining the point and the vertex of the angle bisects the angle.

SUMMARY OF RESULTS

Two triangles are congruent if the following corresponding parts of the triangles are equal:—

(1) Two sides and the included angle.

(2) Two angles and a side.

(3) Three sides.

(4) The hypotenuse and one side (when the triangles are right-angled).

Two triangles are **not necessarily** congruent (that is to say, they may or may not be actually congruent but **we have no right to assume** that they are) if the following corresponding parts are equal.

(1) Two sides and an angle which is not the included angle.

(2) Three angles.

Compare the above with the summary of results in the Construction of Triangles (p. 69).

MISCELLANEOUS RIDERS

In solving riders attention should be paid to the following points :—

1. It is advisable first to draw a rough freehand figure and mark on it the given equal parts. You can then draw a neater and more accurate figure ; this should be as large as is convenient.

2. Make use of all the facts given in the data. For example, if you are given that two straight lines are parallel, you will probably require this fact to prove that certain angles are equal (corresponding or alternate) or supplementary (interior angles on the same side of the cutting line).

3. To prove two lines equal, see whether

(i) they are corresponding sides of two triangles which can be proved to be congruent ;

(ii) they are the sides opposite two equal angles of a triangle.

4. To prove two angles equal, see whether

(i) they are vertically opposite ;

(ii) they are alternate or corresponding angles, formed when a line cuts two parallel lines ;

(iii) they are the third angles of two triangles in which the sum of two angles of the one equals the sum of two angles of the other ;

(iv) they are corresponding angles of two triangles which can be proved to be congruent ;

(v) they are the complements or supplements of two equal angles.

5. Working backwards often provides a key to the solution. Assume the truth of what is to be proved and find what other facts follow from this assumption.

6. To establish a proof it is frequently necessary to join certain points or produce certain lines.

7. Give reasons for your statements. For example, '$\angle AOC = \angle BOD$, because they are vertically opposite.' It is not necessary to quote the number of the theorem in the book. Thus, do not say '$\angle AOC = \angle BOD$ [Theorem 3].'

Example 1. ABC is a triangle in which $\angle B = \angle C$. If A be joined to D, the middle point of BC, prove that AD is perpendicular to BC.

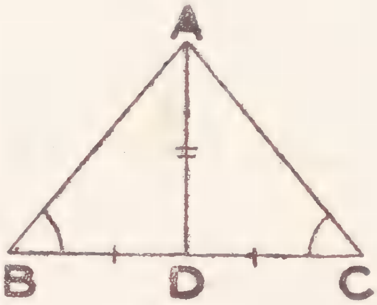


FIG. 113.

In the triangles ABD, ACD, we have $\angle B = \angle C$, $BD = CD$, and AD common. Can we therefore say that the triangles are congruent? No, because the angle is not the **included** angle. We require something further. This is supplied by the fact that, since $\angle B = \angle C$, the side AB = side AC. Hence, to prove the triangles congruent we can use either the '3 sides' theorem [$AB = AC$, $BD = DC$, AD common] or the '2 sides and included angle' theorem [$AB = AC$, $BD = CD$ and $\angle ABD = \angle ACD$]. Having proved the triangles congruent, $\angle ADB = \angle ADC$. But the sum of these two adjacent angles is two right angles. ∴ each is a right angle, that is, AD is perpendicular to BC. The proof can be written out as follows:—

In the $\triangle ABC$, $\angle B = \angle C$.

∴ $AB = AC$.

In the triangles ABD, ACD.

$AB = AC$.

$BD = CD$

included $\angle ABD$ = included $\angle ACD$.

∴ the triangles are congruent.

∴ $\angle ADB = \angle ADC$.

But, because AD is a straight line standing on the straight line BC.

$\angle ADB + \angle ADC = 2$ right angles.

∴ each of these angles is a right angle.

∴ AD is perpendicular to BC.

Example 2. In a quadrilateral ABCD, $AB = CD$ and $\angle B = \angle C$; prove that AD is parallel to BC.

Produce BA and CD to meet at E.

In the Δ EBC

$\angle B = \angle C$

$\therefore EB = EC$.

But $AB = DC$.

$\therefore EA = ED$.

In the Δ EAD, because $EA = ED$

$\therefore \angle EAD = \angle EDA$.

But $\angle EAD + \angle EDA + \angle E = 2$ right angles, being the sum of the angles of a triangle.

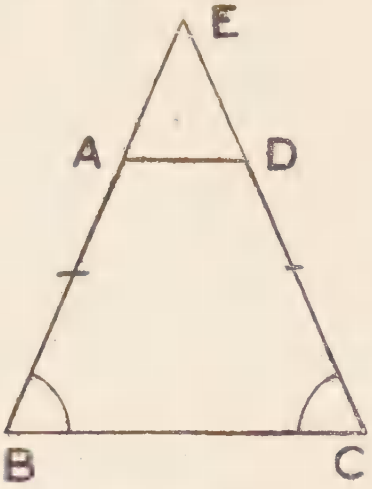


FIG. 114.

$\therefore 2\angle EAD + \angle E = 2$ right angles.

Again, $\angle B + \angle C + \angle E = 2$ right angles being the sum of the angles of a triangle.

But $\angle B = \angle C$.

$\therefore 2\angle B + \angle E = 2$ right angles.

$\therefore 2\angle EAD + \angle E = 2\angle B + \angle E$.

$\therefore \angle EAD = \angle B$.

and these are corresponding angles.

$\therefore$ AD is parallel to BC.

EXERCISE 30

1. ABC is a triangle in which $AB = AC$. P and Q are any points in BC such that $BP = CQ$. Prove that $AP = AQ$.

2. Through the middle point M of a straight line AB any straight line is drawn, and perpendiculars AP, BQ are drawn to it from A, B. Prove that $AP = BQ$.

3. From any point P on the bisector of an $\angle BAC$, PD is drawn parallel to BA meeting AC in D. Prove that Δ APD is isosceles.

4. AOB is a diameter of a circle whose centre is O, and OC is any radius. Prove that the $\angle AOC$ is double the $\angle OBC$.

5. Prove that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

6. ABC is a triangle in which $AB = AC$. BQ and CR bisect the angles B and C and meet the opposite sides in Q and R. Prove that $BQ = CR$.

7. In a triangle ABC, $AB = AC$; BC is produced to D and $\angle ACD$ is double of $\angle B$. Prove that the triangle ABC is equilateral.

8. ABC is a triangle in which AB = AC. AB and AC are produced to P and Q and the bisectors of the angles PBC, QCB meet at O. Prove that (i) OB = OC, (ii) OA bisects ∠ BAC.

9. AB and CD are equal and parallel straight lines. Prove that AD and BC are bisected at their point of intersection.

10. BM, CN are perpendiculars drawn from B, C to the sides AC, AB of a triangle ABC. If BM = CN prove that AB = AC.

11. O is the middle point of a straight line AB. If O be joined to a point C such that OC = OA, prove that ACB is a right angle.

12. AB is a given straight line and P, Q two points outside the line such that PA = PB and QA = QB. Prove that PQ (produced if necessary) bisects AB at right angles.

13. The sides of a quadrilateral ABCD are all equal; prove (i) that the diagonal AC bisects the angles BAD, BCD, (ii) that AB is parallel to CD, and AD parallel to BC.

14. ✓ In a quadrilateral ABCD, AD = BC and AB = CD; prove that AD is also parallel to BC, and AB parallel to CD. [Join the diagonal AC.]

15. O is the middle point of a straight line AB. Through A, B a pair of parallel straight lines are drawn. If through O any line XY be drawn to meet these parallel lines in X and Y, prove that XY is bisected at O.

16. In a quadrilateral ABCD, AB is parallel to DC, and AD parallel to BC. Prove that AB = DC, and AD = BC. [Join AC.]

CHAPTER IX

CONSTRUCTIONS (BY RULER AND COMPASSES)

§ 1. In the constructions of the preceding chapters the ruler has been used for measuring lengths and the protractor for measuring angles. In the constructions of this chapter no such measurements are allowed. The ruler is to be used for two purposes only, namely, (1) to draw a straight line joining two points, and (2) to produce a straight line when necessary. Compasses are used (1) to draw circles, and (2) to cut off given lengths from a straight line.

Each construction can be proved theoretically to be correct, the proof depending on the theorems which have already been established.

CONSTRUCTION 1

§ 2. **Through a given point in a given straight line to draw a straight line making with the given line an angle equal to a given angle.**

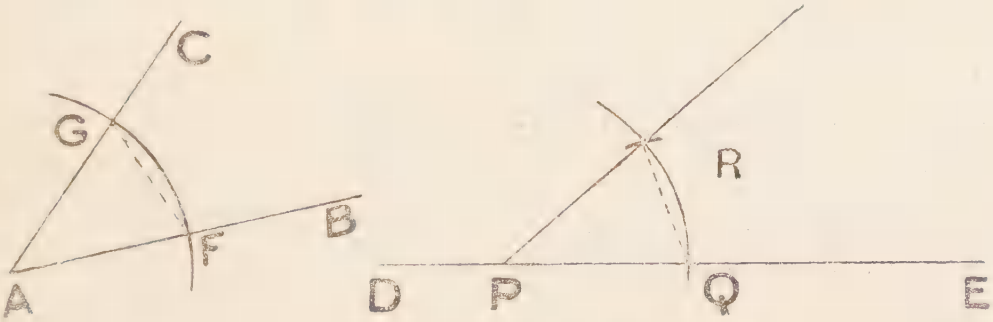


FIG. 115.

Let BAC be the given angle, DE the given straight line and P the given point in it; it is required to draw through P a straight line making with DE an angle equal to the angle BAC.

Construction. With centre A and any radius draw an arc of a circle cutting AB in F and AC in G.

With centre P and the same radius draw an arc cutting DE in Q.

With centre Q and radius equal to FG draw an arc cutting the former arc in R.

Join PR.

Then QPR is the required angle.

Proof. In the triangles PQR, AFG

PQ = AF, being radii of equal circles,
PR = AG, ,, ,, ,,
and QR = FG, ,, ,, ,,
∴ the triangles are congruent.
∴ $\angle QPR = FAG$.

Ex. 1. Draw with the protractor an angle of 48°. Then at a given point construct with ruler and compasses only an angle equal to this angle. Verify, by means of the protractor, the accuracy of your construction.

Repeat this exercise with different angles, acute and obtuse.

Ex. 2. Draw any acute angle. Construct an angle which is double this angle.

Ex. 3. Draw any triangle ABC. At a point O draw consecutive lines OD, OE, OF, OG such that $\angle DOE = \angle A$, $\angle EOF = \angle B$, $\angle FOG = \angle C$.

What fact do you know about OG and OD?

Ex. 4. Draw any quadrilateral ABCD. At a point O draw consecutive lines OP, OQ, OR, OS, OT such that $\angle POQ = \angle A$, $\angle QOR = \angle B$, $\angle ROS = \angle C$, $\angle SOT = \angle D$.

Where should OT lie?

CONSTRUCTION 2

§ 3. To bisect a given angle.

Let BAC be the given angle; it is required to bisect it.

Construction. With centre A and any radius draw an arc of a circle cutting AB in D and AC in E.

With centres D and E draw two circles of the same radius (taking any convenient length for the radius) cutting each other at F.

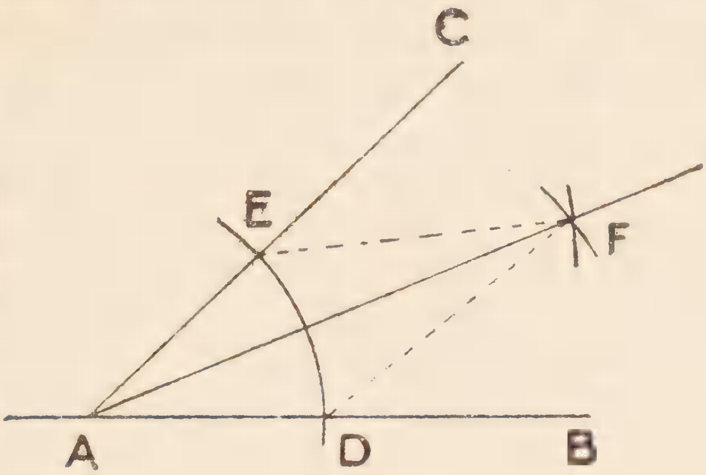


FIG. 116.

Join AF.

Then AF bisects the angle BAC.

Proof. Join DF and EF.

In the triangles ADF, AEF

AD = AE, being radii of the same circle,

DF = EF, being radii of equal circles,

and AF is common.

∴ the triangles ADF, AEF are congruent.

∴ ∠DAF = ∠EAF ;

that is, AF bisects the angle BAC.

Ex. 1. Draw any angles, acute or obtuse. Draw freehand the bisectors of these angles and test your accuracy by constructing the bisectors as above.

Ex. 2. Draw any angle and divide it into four equal angles.

Ex. 3. Draw any acute-angled triangle ; bisect its three angles and produce the bisectors to meet.

[*N.B.*—In exercises where the dimensions are not given figures of a fairly large size should be drawn. It is a common fault with beginners to draw too small a figure.]

Ex. 4. Draw any obtuse-angled triangle ; bisect its three angles and produce the bisectors to meet.

What fact is suggested by this and the preceding exercises?

Ex. 5. Draw any △ ABC and produce the sides AB, AC to D and E. Bisect the ∠BAC and also the exterior angles DBC and ECB. What fact is suggested?

CONSTRUCTION 3

§ 4. To bisect a given straight line.

Let AB be the given straight line ; it is required to bisect it.

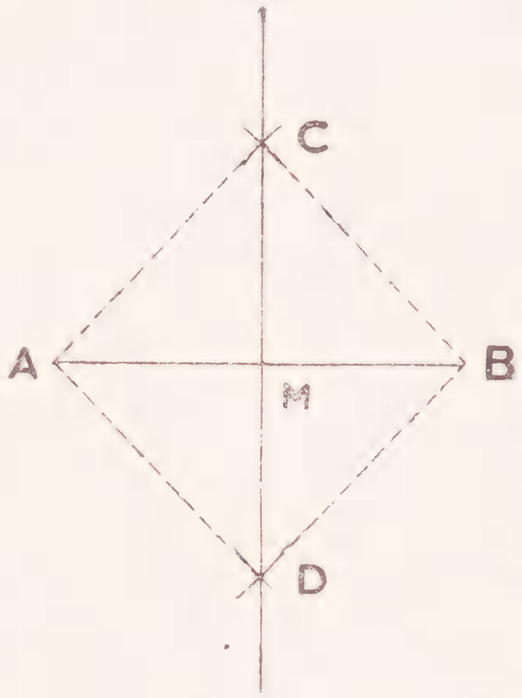


FIG. 117.

Construction. With centres A and B draw two circles of the same radius (any convenient radius) cutting each other in C and D.

Join CD cutting AB in M.

Then AB is bisected at M.

Proof. In the triangles ACD, BCD.

AC = BC, being radii of equal circles.

AD = BD, being radii of equal circles.

CD is common.

∴ the triangles are congruent.

∴ $\angle$ ACD = $\angle$ BCD.

Again, in the triangles ACM, BCM

AC = BC, being radii of equal circles.

CM is common.

and, the included $\angle$ ACM = included $\angle$ BCM (just proved).

∴ the triangles are congruent.

∴ AM = MB.

that is, AB is bisected at M.

Note.—In the above proof it is shown that the triangles ACM, BCM are congruent.

∴ $\angle$ AMC = $\angle$ BMC.

But the sum of these two angles is two right-angles.

∴ each is a right-angle.

Hence the line CD not only bisects AB but is also perpendicular to it.

The line CD is called the **perpendicular bisector** of AB.

Ex. 1. Draw a straight line of some definite length and guess the position of its middle point. Verify by the above construction.

Ex. 2. Draw a straight line of some definite length and divide it into four equal parts.

Ex. 3. The straight line which joins a vertex of a triangle to the middle point of the opposite side is called a **median.**

Draw any triangle and find by the above construction the middle points of the sides; then draw the medians of the triangle.

What fact is suggested?

Ex. 4. Draw (i) an acute-angled triangle, (ii) a right-angled triangle and (iii) an obtuse-angled triangle. Draw the perpendicular bisectors of the three sides.

What fact is suggested?

CONSTRUCTION 4

§ 5. **To draw a straight line perpendicular to a given straight line from a point in it.**

Let AB be the given straight line and P the given point in it; it is required to draw from P a straight line perpendicular to AB.

Construction. With centre P cut off from AB equal parts PC, PD.

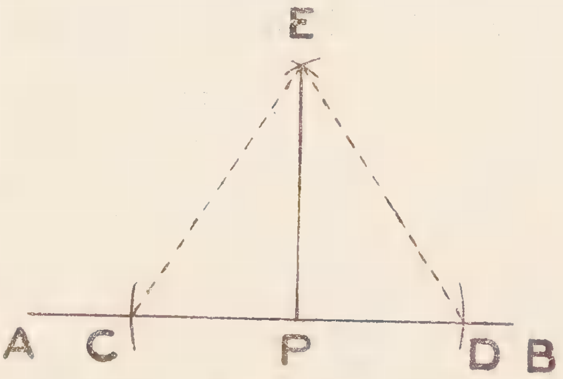


FIG. 118.

With centre C and any radius greater than CP draw an arc of a circle.

With centre D and the same radius draw an arc cutting the former arc at E.

Join PE.

Then PE is perpendicular to AB.

Proof. In the triangles ECP, EDP

PC = PD.

CE = DE, radii of equal circles

and PE is common.

∴ the triangles are congruent.

∴ ∠EPC = ∠EPD.

But the sum of these two angles is two right angles.

∴ each is a right angle.

∴ PE is perpendicular to AB.

Ex. 1. Draw a straight line and from any point in it, construct a straight line at right angles to it. Test with set square.

Note.—When the given point is at the end of the given line it is necessary to produce the line.

Ex. 2. Use the above construction to draw a square each side of which is 8 cm. long.

Ex. 3. Draw a rectangle with a pair of adjacent sides 3·4″ and 2·1″ long.

CONSTRUCTION 5

§ 6. **To draw a straight line perpendicular to a given straight line from a given point outside it.**

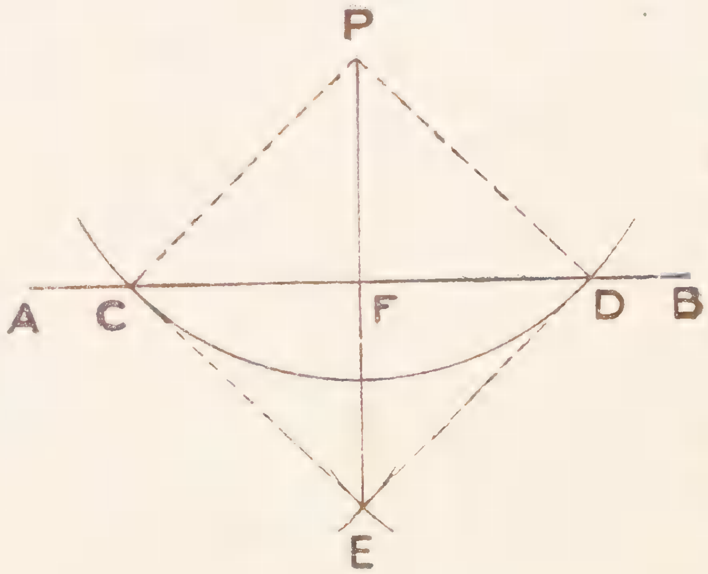


FIG. 119.

Let AB be the given straight line and P the given point outside it; it is required to draw from P a straight line perpendicular to AB.

Construction. With centre P and any convenient radius draw an arc of a circle cutting AB in the points C and D.

With centres C and D draw two circles of the same radius (any convenient radius) cutting each other at E.

Join PE cutting CD in F.

Then PF is perpendicular to AB

Proof. Join CP, DP, EC, ED.

In the triangles PCE, PDE

PC = PD, radii of the same circle.

CE = DE, radii of equal circles.

and PE is common.

∴ the triangles are congruent.

∴ ∠CPE = ∠DPE.

Again, in the triangles PCF, PDF

PC = PD.

PF is common.

and the included ∠CPF = included ∠DPF.

∴ the triangles are congruent.

∴ ∠PFC = ∠PFD.

But the sum of these two angles is two right angles.

∴ each is a right angle.

that is, PF is perpendicular to AB.

Note.—In practical exercises it is more convenient to use set squares for drawing perpendiculars rather than the method of Constructions 4 and 5. Perpendiculars should be drawn by means of these constructions only when the use of set squares and protractors is forbidden.

Ex. Draw (i) any acute-angled triangle, (ii) any obtuse-angled triangle, and from its vertices draw, by means of Construction 5, perpendiculars to the opposite sides (produced if necessary).

What fact is suggested ?

CONSTRUCTION OF SPECIAL ANGLES

§ 7. The foregoing constructions enable us to construct certain angles by means of ruler and compasses only.

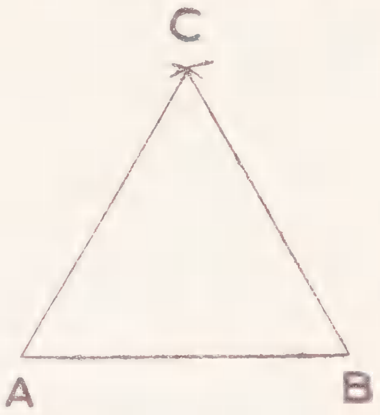


FIG. 120.

(1) **To construct an angle of 60°.**

Draw any equilateral triangle ABC. From Theorem 16 it follows that, since the sides are all equal, the angles are all equal. But the sum of the angles is 180°.

∴ each angle of the △ABC is 60°.

(2) **To construct an angle of 30°.**

Construct an angle of 60° and bisect it.

(3) **To construct an angle of 90°.**

This is the same as drawing a straight line perpendicular to another straight line. Therefore either Construction 4 or 5 may be used.

(4) **To construct an angle of 45°.**

Construct an angle of 90° and bisect it. Angles which can be expressed as multiples or sub-multiples, or as the sum or difference, of the above angles can be constructed by similar methods.

Ex. 1. Construct angles of (i) 15°, (ii) $22\frac{1}{2}°$.
Ex. 2. ,, ,, ,, (i) 120°, (ii) 150°, (iii) 135°.
Ex. 3. ,, ,, ,, (i) 75°, (ii) 105°.
Ex. 4. Trisect (that is, divide into 3 equal parts) :—
(i) a right angle, (ii) an angle of 45°.

§ 8. The fact established in Theorem 8 regarding alternate angles provides us with a method of constructing through a given point a straight line parallel to a given straight line.

Thus, in Fig. 121, let P be the given point and AB the given straight line. Draw any line PC to meet AB in C. By means of Construction 1 draw an ∠FPG = ∠ECD. Then, since these are alternate angles PG is parallel to AB.

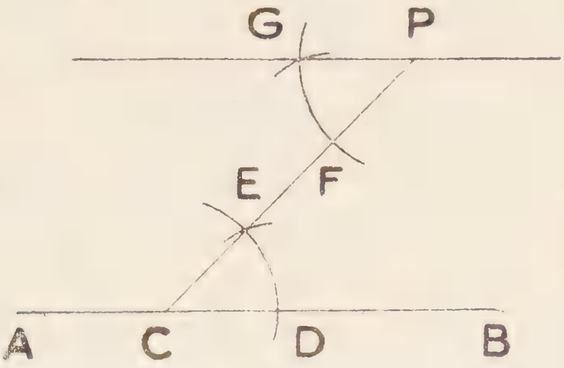


FIG. 121.

In practice, instead of taking any length CD as the radius, it is more convenient to take the whole line CP. As follows :

CONSTRUCTION 6

Through a given point to draw a straight line parallel to a given straight line.

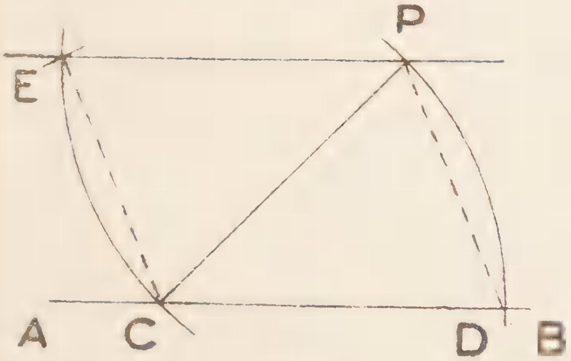


FIG. 122.

Let P be the given point and AB the given straight line ; it is required to draw through P a straight line parallel to AB.

Construction. Join P to any point C in AB.

With centre C and radius CP draw an arc of a circle cutting AB in D.

With centre P and the same radius draw an arc of a circle passing through C.

With centre C and radius equal to DP draw an arc of a circle cutting the latter arc in E.

Join PE.

Then PE is parallel to AB.

Proof. Join CE and PD.

In the triangles **PCD, CPE**
CD = **PE**, radii of equal circles
PD = **CE**, ,, ,, ,,
and **CP** is common.
∴ the triangles are congruent.
∴ ∠**PCD** = ∠**CPE**.
But these are alternate angles.
∴ **PE** is parallel to **AB**.

Ex. 1. Through a given point draw a parallel to a given straight line by making a pair of corresponding angles equal.

Ex. 2. Draw any two straight lines **AB**, **BC**, meeting at **B**. Use Construction 6 to draw through **A** a straight line parallel to **BC** and through **C** a straight line parallel to **BA**. Let these straight lines meet at **D**. Measure the sides and angles of the figure **ABCD**. What do you observe?

Ex. 3. Draw any triangle **ABC**. Through each vertex draw, by Construction 6, a straight line parallel to the opposite side. Let these straight lines meet to form a new triangle. Measure the angles of this new triangle and also the angles of the triangle **ABC**. What do you observe?

TEST PAPERS

I

1 Explain what is meant by an angle ?

What angle does the hour-hand of a clock turn through in one hour ?

2. Draw a straight line **AB** 7 cm. long. Find a point which is 4·5 cm. from **A** and 5 cm. from **B**. How many such points are there ?

3. From a point **O** in a straight line **AB** two straight lines **OC**, **OD** are drawn on the same side of **AB** such that ∠ **AOC** = 57° and ∠ **COD** = 83°. What is ∠ **DOB** ?

4. **ABC** is any triangle and **AD** a line drawn to any point **D** in the side **BC**. If ∠ **ADC** = 127°, ∠ **ABC** = 63° and ∠ **DAC** = 25° find all the angles of the figure.

5. A ship sails 10 miles East, then 18 miles South-west and then 25 miles North-west. Draw a plan of her course to any convenient scale and find from it her distance from the starting-point.

6. When one straight line stands on another straight line what is the sum of the adjacent angles ?

Use this fact to prove that when two straight lines intersect the vertically opposite angles are equal.

II

1. Explain in what way straight lines differ from curved lines.

2. Draw a triangle **ABC** with ∠ **A** = 72°, **AB** = 3·2″ and **AC** = 2·4″. Draw a straight line from **C** perpendicular to **AB** and measure its length.

3. What are parallel straight lines ?

When a straight line cuts two parallel straight lines what do you know about the angles so formed ?

4. Find the sum of the angles of a polygon of 13 sides.

5. **ABC** is any triangle and **D** any point in the side **BC**. From **D** a line **DE** is drawn parallel to **BA** meeting **AC** in **E**. If ∠ **ABC** = 44°, ∠ **ADE** = 50° and ∠ **DAE** = 38°, find all the other angles of the figure, giving reasons.

6. **AOB**, **COD** are two straight lines intersecting at **O**; **OR**, **OS** are the bisectors of the angles **AOC**, **BOD**. Prove that **OR** and **OS** are in the same straight line.

III

1. Define right angle, obtuse angle, supplementary angles.

2. Draw an ∠ **ABC** = 120°, making **AB** = 6·5 cm. and **BC** = 5·2 cm. With ruler and set square draw through **A** a line parallel to **BC** and through **C** a line parallel to **BA**. Let these lines meet in **D**. Measure ∠ **D** and the sides **AD**, **CD**.

3. What is the sum of the angles of a quadrilateral?

ABCD is a quadrilateral and the side **BC** is produced to **E**. If ∠ **ECD** = 35°, ∠ **ADC** = 107°, and ∠ **DAB** = 68°, find ∠ **ABC**.

4. The angle of elevation of the top of the pinnacle of the Taj Mahal as seen from a certain point on the ground is 28°. On walking a distance of 150 ft. towards it along the level ground the angle of elevation is found to be 40°. Draw a plan to a scale of 1 inch to 100 ft. and find from it the height of the top of the pinnacle above the ground.

5. **PQR** is any triangle and the side **QR** is produced to **S**. Prove, by drawing a line through **R** parallel to **QP**, that the exterior angle **PRS** is equal to the sum of the interior opposite angles **P** and **Q**.

6. Show how to find the number of degrees in an angle of a regular hexagon.

Construct a regular hexagon of side 5 cm.

IV

1. Through what angle does the minute-hand of a clock turn between (i) 10.15 p.m. and 10.25 p.m.; (ii) 6.10 p.m. and 6.35 p.m.?

2. **LM** cuts a pair of parallel straight lines **AB**, **CD** in **R**, **S**. Name (i) a pair of alternate angles, (ii) a pair of corresponding angles.

If ∠ **RSD** = 60°, what are all the other angles?

3. Draw a triangle **PQR** in which **QR** = 7·8 cm., ∠ **Q** = 58°, and ∠ **P** = 82°. Explain the steps of your construction.

Draw, with ruler and set square, perpendiculars from the angular points to the opposite sides. Measure the length of each perpendicular.

4. Patna is 288 miles N.W. of Calcutta. Mymensingh is 336 miles from Patna and 192 miles from Calcutta and eastward of

both towns. Find the bearing of Mymensingh from (i) Calcutta, (ii) Patna. [Scale 1″ to 80 miles.]

5. Explain what is meant by saying that two triangles are congruent.

Are two triangles congruent if the following corresponding parts of the triangles are equal? (i) 3 sides, (ii) 3 angles, (iii) 2 sides and one angle, (iv) 2 angles and one side.

6. ABCD is a quadrilateral in which AB = CD and AD = BC and all the angles are right angles. Prove that AC = BD.

V

1. Define isosceles triangle, right-angled triangle, hypotenuse.

2. Two angles of a triangle are 76° and 84°; what is the third angle?

3. Construct a rectangle with one side 2·7 in. and a diagonal 3·3 in. Explain the method of construction.

4. From the top of a cliff 360 ft. above sea-level a person observes two boats due East of him and finds their angles of depression to be 33° and 13°. Draw a plan to a scale of 1 in. to 400 ft. and find from it the approximate distance between the two boats.

5. Prove that if two angles of a triangle are equal the sides opposite those angles are equal.

6. ABC is any triangle. The bisector of the angle A meets BC in D. Through D a line DE is drawn parallel to AB meeting AC in E. Prove that AE = ED.

VI

1. Find the number of degrees in the vertical angle of an isosceles triangle in which each of the base angles is 4 times the vertical angle.

2. Construct a triangle ABC in which BC = 4·3 cm., CA = 5·2 cm., AB = 7·5 cm.

With ruler and compasses only draw perpendiculars from each vertex to the opposite side (produced if necessary); measure the length of each perpendicular.

3. Mysore is 250 miles S. 75° W. of Madras, and Trichinopoly is 180 miles S. 54° E. of Mysore. Find the distance and bearing of Trichinopoly from Madras.

4. What fact do you know about corresponding angles?

Use this fact to draw through a given point a straight line parallel to a given straight line.

5. Prove that two right-angled triangles are congruent if the hypotenuse and one side of the one are respectively equal to the hypotenuse and one side of the other.

6. ABC is an isosceles triangle with AB = AC. Points P, Q are taken in BC produced on opposite sides of the triangle such that BP = CQ. Prove that (i) the triangles APB, AQC are congruent, (ii) the triangle APQ is isosceles.

VII

1. Explain the terms vertically opposite angles, adjacent angles, complementary angles.

2. Construct a quadrilateral ABCD given AB = 2·1″, BC = 2·9″, CD = 2·7″, diagonal AC = 3″, diagonal BD = 3·5″. Measure the angles of the quadrilateral and the remaining side AD.

3. What is the sum of the angles of a triangle?

Draw an isosceles triangle with vertical angle A = 52″ and base BC = 2·2″. State the steps of your construction.

4. A steamer sailing at the rate of 14 miles an hour sails for one hour in a direction S. 25° W., then for half an hour in a direction S. 80° W. and finally for one hour and a half in a direction S. 15° E. Draw a plan of her course to scale and find from it her distance from the starting-point.

5. Show how to bisect an angle by ruler and compasses only.

Prove that your method of construction is correct.

6. A straight line XY meets two parallel straight lines AB, CD in EF. The bisectors of the angles BEF, EFD meet in G. Prove that ∠EGF is a right angle.

VIII

1. Define quadrilateral, square, rectangle.

2. Draw a triangle ABC in which AB = 1·6″, BC = 3·4″ and ∠B = 35°. Measure the remaining side and angles.

Draw the perpendicular bisectors of the sides, using ruler and compasses.

3. How many degrees are there in an angle of an equilateral triangle?

ABC, ACD are equilateral triangles. Prove that AD is parallel to BC, and AB parallel to CD.

4. From a point on the **maidan** at Calcutta I find the angle of elevation of the top of the Victoria Memorial to be 20°. On

walking 100 yards towards it I find the angle of elevation to be 45°. Draw a plan to scale and find from it the height of the Memorial in feet.

5. What is a regular polygon ? Calculate the number of degrees in an angle of a regular polygon of (i) 8 sides, (ii) 40 sides.

6. Prove that the sum of the angles of a triangle is two right angles.

IX

1. Explain the terms vertical, horizontal, angle of elevation, angle of depression.

2. ABCD is a quadrilateral in which AD is parallel to BC ; the diagonals AC, BD meet in E. If ∠ ABC = 74°, ∠ AEB = 82°, ∠ BCD = 80°, and ∠ ADB = 42°, find all the other angles of the figure, giving reasons.

3. Construct, with ruler and compasses only, angles of 30° and 75°.

4. Draw a triangle ABC, given AB = 2·8″, AC = 1·9″, ∠ B = 40°.

Can you draw more than one kind of triangle to contain the given parts ?

5. Prove that the sum of the angles of a polygon of n sides is $(2n - 4)$ right angles.

How many sides has a regular polygon each angle of which is 160° ?

6. OA, OB are the equal arms of an angle AOB. Points C, D are taken in OA, OB such that OC = OD. Prove (i) the triangles OAD, OBC congruent, (ii) the triangles EAC, EBD congruent, E being the point of intersection of AD and BC.

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407N. Annamalai Kottaiyur

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